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Dimensional Reduction of Supersymmetric Gauge Theories

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Abstract

Main objective of the present dissertation is the determination of all the possible low energy models which emerge in four dimensions by the dimensional reduction of a gauge theory over multiple connected coset spaces. The higher dimensional gauge theory is chosen to be the one that the Heterotic string theory suggests: (i) it is defined in ten dimensions, (ii) it is based on the $E_8 \times E_8$ symmetry group and (iii) it is $\mathcal{N} = 1$ globally supersymmetric. The search of all four-dimensional gauge theories resulting from the aforementioned dimensional reduction, is restricted only to models which are potentially interesting from a phenomenological point of view. This requirement constrain these models to come from one of the known Grand Unified Theories (GUTs) in an intermediate stage of the spontaneous symmetry breaking. Main result of my study is that extensions of the Standard Model (SM) which are based on the Pati-Salam group structure can be obtained in four dimensions.

A second direction of research which is discussed in this dissertation is based on the following conclusions of a previous research work: (i) It is possible to obtain four-dimensional theories with a non-abelian gauge symmetry by the dimensional reduction of a higher dimensional $U(1)$ noncommutative theory and (ii) the particle physics models resulting from this particular dimensional reduction are renormalizable. The objective of the present dissertation in this direction is the study of the last remark. Starting with the most general renormalizable gauge theory with scalar fields (consistent with the dimensional reduction over a fuzzy sphere) which can be defined in four dimensions, it turns out that fuzzy extra dimensions emerge dynamically. This is supported by the calculation of the spectrum of vector and scalar bosons. In this way, the renormalizability of the four-dimensional low energy models resulting from the dimensional reduction over fuzzy coset spaces is verified.

Finally, it is extremely interesting to assume noncommutative characteristics not only

for the internal space of a higher dimensional theory, but also for the four-dimensional Minkowski space, M^4, providing that these appear only in an elementary length scale which is assumed to be the Planck length. In this framework, the approach of linearized noncommutative gravity was examined and the connection of the algebra describing the noncommutative space with its geometry was studied. It turned out that the linear perturbation which describes the noncommutativity of the algebra contributes to the Ricci curvature tensor in a non-trivial way. This conclusion suggests a possible fundamental connection between the noncommutative geometry and the theory of gravity, which have to be investigated further.

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Contents

Chapter 1

Introduction

The quest for unification of all observed interactions at low energies has been going on for many years. The successful unification of electromagnetic and weak interactions was achieved of the semisimple gauge group $SU(2) \times U(1)$, which was spontaneously broken to $U(1)$ at a scale of $\mathcal{O}(100GeV)$ [1,2]. For the spontaneous symmetry breaking of the theory, a scalar field sector was introduced in an ad hoc manner. An attractive framework to unify the strong and the electroweak interactions is provided by Grand Unified Theories (GUTs), which make use of a simple gauge group, e.g. $SU(5)$, $SO(10)$, E_6 [3–5]. In GUTs a further scale of $\mathcal{O}(10^{15}GeV)$, related to the superstrong symmetry breaking of the theory, had to be introduced in addition to electroweak symmetry breaking scale. Then the scalar sector had to be enlarged even further. In addition a new complication appeared, the so-called hierarchy problem, related to the huge difference between the two scales [6].

Although the GUT unification scale is not very far from the Planck scale, GUTs do not incorporate the gravitational interactions. On the other hand, the earliest unification attempts of Kaluza and Klein [7,8] included gravity and electromagnetism, which were the established interactions at that time. At the heart of the Kaluza-Klein scheme lies the assumption that space-time has more than four dimensions, which has been considered too speculative.

The Kaluza-Klein proposal was to reduce a pure gravity theory from five dimensions to four, which led to a $U(1)$ gauge theory, identified with electromagnetism, coupled to gravity. A revival of interest in the Kaluza-Klein scheme started after the realisation [9, 10] that non-abelian gauge groups appear naturally when one further extend the space-time dimensions. With the assumption that the total space-time manifold can be written as a direct product $M^D = M^4 \times B$, where B is a compact Riemannian space with a non-abelian isometry group S, dimensional reduction of the theory leads to gravity coupled to a Yang-Mills theory with a gauge group containing S and scalars in four dimensions. The main advantage of this picture is the geometrical unification of gravity with the other interactions and also the explanation of gauge symmetries. There are, however, some problems in the Kaluza-Klein framework. One of the most serious obstacle to obtaining

a realistic model of the low-energy interactions seems to be that after adding fermions to the original action it is impossible to get chiral fermions in four dimensions [11]. If, however, one adds suitable matter fields to the original action - in particular Yang-Mills fields - then one can have massless fermions and parity violation in the fermion sector [12, 13]. Thus one is led to introduce Yang-Mills fields in higher dimensions. In fact, in some other popular schemes such as supergravity [14] and superstring theories [15, 16] the Einstein-Yang-Mills theory appears in the bosonic sector.

It is a common belief that the effects of gravity are negligible for low-energy phenomena. Therefore, inasmuch as one is interested in describing only the low-energy interactions, one can take the bold step to neglect gravity altogether, assuming, however, the direct product of space-time $M^D = M^4 \times B$. Then one starts with a Yang-Mills theory defined on $M^D = M^4 \times B$, yielding a Yang-Mills-Higgs theory in four dimensions. This provides a potential unification of low-energy interactions as well as of gauge and Higgs fields. A naive and crude way to fulfil this requirement is to discard the field dependence on the extra coordinates. A more elegant one is to allow for a non-trivial dependence on them, but impose the condition that a symmetry transformation by an element of the isometry group S of the space formed by the extra dimensions B corresponds to a gauge transformation. Then the Lagrangian will be independent of the extra coordinates just because it is gauge invariant. This is the basis of the CSDR scheme [17–19], which assumes that B is a compact coset space, S/R. The requirement that transformations of the fields under action of the symmetry group of S/R are compensated by gauge transformations, leads to certain constraints on the fields.

It is worth recalling that the Coset Space Dimensional Reduction (CSDR) [17–19] was suggesting from the beginning that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. Phenomenologically interesting GUTs which are obtained by the application of the CSDR method have been reported in [18]. However their surviving scalars transform in the fundamental of the resulting gauge group and are not suitable for the superstrong symmetry breaking towards the SM. As a way out to it has been suggested [18, 20] to take advantage of non-trivial topological properties of the extra compactification coset space, apply the Hosotani or Wilson flux breaking mechanism [21–23] and break the gauge symmetry of the theory further [18, 24]. The main objective of my work is the investigation to which extent applying both methods namely CSDR and Wilson flux breaking mechanism, one can obtain reasonable low-energy models.

In chapter 2 I present the CSDR scheme in sufficient detail to make the dissertation self-contained and give a simple example of the method. Namely, I discuss the dimensional reduction of an $\mathcal{N} = 1$ Yang-Mills-Dirac theory over the six-dimensional sphere S^6. In chapter 3 I recall the Wilson flux breaking mechanism and make some important remarks in order to apply the method on models resulting from dimensional reduction. In chapter 4 starting with an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory defined in ten dimensions, I classify the low-energy models resulting from CSDR and a subsequent application of Wilson flux spontaneous symmetry breaking. The space-time on which the theory is defined can be written in the compactified form $M^4 \times B$, with M^4 the ordinary Minkowski

spacetime and $B = S/R$ a six-dimensional homogeneous coset space. I constrain my investigation in those cases that the dimensional reduction leads to phenomenologically interesting and anomaly-free four-dimensional GUTs such as E_6, $SO(10)$. In chapters 5-8 I discuss noncommutative generalisations of the CSDR scheme with emphasis on the renormalizability of the emergent four-dimensional theories. Finally, in chapter 9 I present some aspects of the more adventurous assumption of promoting the ordinary spacetime to a noncommutative 'manifold'.

Chapter 2

Coset Space Dimensional Reduction

The celebrated Standard Model (SM) of Elementary Particle Physics had so far outstanding successes in all its confrontations with experimental results. However the apparent success of the SM is spoiled by the presence of a plethora of free parameters mostly related to the ad-hoc introduction of the Higgs and Yukawa sectors in the theory. It is worth recalling that the Coset Space Dimensional Reduction (CSDR) [17–19] was suggesting from the beginning that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. The four-dimensional gauge and Higgs fields are simply the surviving components of the gauge fields of a pure gauge theory defined in higher dimensions. Here, I describe the CSDR scheme giving emphasis on the construction of symmetric fields and discuss the resulting four-dimensional theory. For various details consult [18, 25] whereas the consistency of the method have been proven in [26].

2.1 Introduction

In the CSDR scheme one assume a gauge theory defined over a higher dimensional space which is compactified in the form of $M^4 \times (S/R)$, where S/R a compact coset space. Basis of the theory is to allow the higher dimensional fields to have a non-trivial dependence on the extra coordinates. This is realised by imposing the condition that a symmetry transformation by an element of the isometry group S of the space formed by the extra dimensions B corresponds to a gauge transformation. Then the Lagrangian will be independent of the extra coordinates just because it is gauge invariant. The theory provides a gauge-Higgs unification with the four-dimensional gauge and Higgs fields to be simply the surviving components of the gauge fields of a pure gauge theory defined in higher dimensions. The introduction of fermions [27] was a major development. Then the four-dimensional Yukawa and gauge interactions of fermions found also a unified description in the gauge interactions of the higher dimensional theory. Recent improvement in this unified description in high dimensions is to relate the gauge and fermion fields that have been introduced [28–30]. A simple way to achieve that is to

demand that the higher dimensional gauge theory is $\mathcal{N} = 1$ supersymmetric which requires that the gauge and fermion fields are members of the same supermultiplet. An additional strong argument towards higher dimensional supersymmetry including gravity comes from the stability of the corresponding compactifying solutions that lead to the four-dimensional theory.

In the spirit described above a very welcome additional input is that string theory suggests furthermore the dimension and the gauge group of the higher dimensional supersymmetric theory [15,16,31]. Further support to this unified description comes from the fact that the reduction of the theory over coset [18] and CY spaces [15,16,31] provides the four-dimensional theory with scalars belonging in the fundamental representation (rep.) of the gauge group as are introduced in the SM. In addition the fact that the SM is a chiral theory lead us to consider D-dimensional supersymmetric gauge theories with $D = 4n + 2$ [13, 18], which include the ten dimensions suggested by the heterotic string theory [15, 16, 31].

Concerning supersymmetry, the nature of the four-dimensional theory depends on the corresponding nature of the compact space used to reduce the higher dimensional theory. Specifically the reduction over CY spaces leads to supersymmetric theories [15,16,31] in four dimensions, the reduction over symmetric coset spaces leads to non-supersymmetric theories, while a reduction over non-symmetric ones leads to softly broken supersymmetric theories [28–30].

In section 2.2 I present the CSDR scheme in sufficient detail to make the dissertation self-contained. I especially discuss some elements of the coset space geometry, the reduction of a higher dimensional Yang-Mills-Dirac action and the constraints that the surviving fields have to fulfil. I finally make some remarks on the four-dimensional Lagrangian. In section 2.3 an example of the method is given. I describe the dimensional reduction of an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory over the six-dimensional sphere $SO(7)/SO(6) \sim S^6$.

2.2 Coset Space Dimensional Reduction

Given a gauge theory defined in higher dimensions the obvious way to dimensionally reduce it is to demand that the field dependence on the extra coordinates is such that the Lagrangian is independent of them. A crude way to fulfil this requirement is to discard the field dependence on the extra coordinates, while an elegant one is to allow for a non-trivial dependence on them, but impose the condition that a symmetry transformation by an element of the isometry group S of the space formed by the extra dimensions B corresponds to a gauge transformation. Then the Lagrangian will be independent of the extra coordinates just because it is gauge invariant. This is the basis of the CSDR scheme [17–19], which assumes that B is a compact coset space, S/R.

In the CSDR scheme one starts with a Yang-Mills-Dirac Lagrangian, with gauge group G, defined on a D-dimensional space-time M^D, with metric g^{MN}, which is compactified

to $M^4 \times S/R$ with S/R a coset space. The metric is assumed to have the form

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{pmatrix}, \tag{2.1}$$

where $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ and g^{ab} is the coset space metric. The requirement that transformations of the fields under the action of the symmetry group of S/R are compensated by gauge transformations leads to certain constraints on the fields. The solution of these constraints provides the four-dimensional unconstrained fields as well as the gauge invariance that remains in the theory after dimensional reduction. Therefore a potential unification of all low-energy interactions, gauge, Yukawa and Higgs is achieved, which was the first motivation of this framework.

It is interesting to note that the fields obtained using the CSDR approach are the first terms in the expansion of the D-dimensional fields in harmonics of the internal space B. The effective field theories resulting from compactification of higher dimensional theories contain also towers of massive higher harmonics (Kaluza-Klein) excitations, whose contributions at the quantum level alter the behaviour of the running couplings from logarithmic to power [32]. As a result the traditional picture of unification of couplings may change drastically [33]. Higher dimensional theories have also been studied at the quantum level using the continuous Wilson renormalisation group [34, 35] which can be formulated in any number of space-time dimensions with results in agreement with the treatment involving massive Kaluza-Klein excitations.

2.2.1 Coset space geometry

Here I recall some aspects of the coset space geometry. One can divide the generators of S, Q_A in two sets: the generators of R, Q_i $(i = 1, \ldots, \dim(R))$, and the generators of S/R, $Q_a(a = \dim R + 1 \ldots, \dim(S))$, and $\dim(S/R) = \dim(S) - \dim(R) = d$. Then the commutation relations for the generators of S are the following

$$[Q_i, Q_j] = f_{ij}{}^k Q_k \,, \tag{2.2a}$$
$$[Q_i, Q_a] = f_{ia}{}^b Q_b \,, \tag{2.2b}$$
$$[Q_a, Q_b] = f_{ab}{}^i Q_i + f_{ab}{}^c Q_c \,. \tag{2.2c}$$

So S/R is assumed to be a reductive but in general non-symmetric coset space. When S/R is symmetric, the $f_{ab}{}^c$ in (2.2c) vanish.

The above splitting of the S generators can be characterised by determining the decomposition of the adjoint rep. of S under R.

$$S \supset R \tag{2.3}$$
$$\mathrm{adj}(S) = \mathrm{adj}(R) + \mathrm{v}\,, \tag{2.4}$$

where v corresponds to the coset generators.

The tangent space vectors transform under rotations of $SO(d)$ among themselves. Actually the generators Q_a form a vector on the coset space, so they transform as the vector rep. of $SO(d)$, d. On the other hand, as it was stated above, the Q_a generators transform also as v of R, thus there must be an embedding of R into $SO(d)$ such that $d = \text{v}$. This embedding is determined as soon as the embedding of R into S is made. Consider the $SO(d)$ commutation relations

$$[\Sigma^{ab}, \Sigma^{cd}] = -g^{ac}\Sigma^{bd} + g^{ad}\Sigma^{bc} + g^{bc}\Sigma^{ad} - g^{bd}\Sigma^{ac}\,, \tag{2.5}$$

where Σ^{ab} are the $SO(d)$ generators and g^{ab} is the metric of the tangent space. Now the embedding of R into $SO(d)$ is determined by

$$T_i = -\frac{1}{2} f_{iab}\Sigma^{ab}\,, \tag{2.6}$$

since one can show that the T_i form an R-subalgebra of $SO(d)$ using the Jacobi identities and the commutation relations (2.5).

Let me now introduce the coordinates on the $M^4 \times (S/R)$ space $x^M = (x^\mu, y^\alpha)$. In order to parametrise the coset I choose a representative element

$$L(y) = exp(y^\alpha \delta^a_\alpha Q_a)\,, \tag{2.7}$$

of each R-equivalence class. I use greek indices to denote the coordinates of the coset space and latin ones for coordinates of the tangent space. The δ^a_α is used to connect the indices of the manifold and those of the tangent space. Consider the action of an S-transformation, s, on the representative element $L(y)$*. This will give another element of S, which in general will belong to a different equivalent class, whose representative I denote by $L(y')$. Then an extra transformation $r \in R$ is needed to bring $L(y')$ to that element of S. This can be expressed as

$$L(y)s = r(y,s)L(y')\,. \tag{2.8}$$

The equation determines both y' and r as a function of y and s. We use ω^i and ω^α to parametrise S and ϕ^i to parametrise R, i.e., $s = exp(\omega^i Q_i + \omega^\alpha \delta^a_\alpha Q_a)$ and $r = exp(\phi^i Q_i)$. If I consider an infinitesimal S-transformation, s, in the neighbourhood of the identity I obtain

$$\delta y^\alpha = y'^\alpha - y^\alpha = \omega^\beta \delta^a_\beta \xi^\alpha_a + \omega^i \xi^\alpha_i\,, \tag{2.9}$$

where ξ^α_a and ξ^α_i are vector fields (they are in fact Killing vector fields as will be shown later), tangential in the direction of the given transformation. The infinitesimal R-transformations to order ω are

$$\phi^i = \omega^\beta \delta^a_\beta \Omega^i_a + \omega^j \xi^i_j\,. \tag{2.10}$$

The coefficients Ω^i_a and Ω^j_i are sometimes called R-compensators, as they are associated with the compensating R-transformation.

*We assume right cosets only.

Now inserting eqs (2.9), (2.10) into eq. (2.8) I find that the Killing vectors and the R-compensators are given as

$$\xi_a^\alpha = \delta_a^\alpha - \frac{1}{2} y^\beta \delta_\beta^b f_{ab}{}^c \delta_c^\alpha + \ldots , \qquad \xi_i^\alpha = -y^\beta \delta_\beta^a \delta_c^\alpha f_{ia}{}^c + \ldots , \tag{2.11}$$

$$\Omega_j^i = \delta_j^i + \ldots , \qquad \Omega_\alpha^i = -\frac{1}{2} y^\beta \delta_\beta^b \delta_\alpha^a f_{ab}{}^i + \ldots . \tag{2.12}$$

It is evident then that at $y = 0$, $\xi_a^\alpha = \delta_a^\alpha$, $\xi_i^\alpha = 0$, $\Omega_j^i = \delta_j^i$ and $\Omega_\alpha^i = 0$, a fact I shall use later. Furthermore the Killing vectors, found in (2.11), obey the algebra

$$\xi_A^\beta \partial_\beta \xi_B^\alpha - \xi_B^\beta \partial_\beta \xi_A^\alpha = f_{AB}{}^C \xi_C^\alpha . \tag{2.13}$$

The vielbein and the R-connection are defined through the Maurer-Cartan form which takes values in the Lie algebra of S

$$L^{-1}(y) dL(y) = e(y) = e_\alpha^a Q_a + e^i Q_i . \tag{2.14}$$

where $e(y)$ is a Lie algebra valued one-form. Let me recall here some properties of the exterior derivative d; d is a linear operator satisfying

(i) for a function f, $df = (\partial_\alpha f) dy^\alpha = (\partial_\alpha f) e^\alpha$,

(ii) $d^2 = 0$,

(iii) for a q-form u, $d(u \wedge v) = du \wedge v + (-1)^q u \wedge dv$.

The one-form $e(y)$ obeys the so called Maurer-Cartan equations

$$de(y) = e(y) \wedge e(y) , \tag{2.15}$$

which can be easily proved by using eq. (2.14) and the properties of d. Using now the commutation relations (2.2c) and (2.14) I can rewrite eq. (2.15) in terms of R and S/R components of the one-form $e(y)$ as

$$de^a = \frac{1}{2} f_{bc}{}^a e^b \wedge e^c + f_{bi}{}^a e^i , \qquad de^i = \frac{1}{2} f_{ab}{}^i e^a \wedge e^b + \frac{1}{2} f_{jk}{}^i e^j \wedge e^k . \tag{2.16}$$

The Maurer-Cartan equations are very useful to calculate various quantities relevant to characterise the geometry of the manifold, such as the connection and the curvature. Using the parametrisation of the coset (2.7) and the components of the one-form $e(y)$ can be easily computed, for infinitesimal y, to be

$$e_\alpha^a(y) = \delta_\alpha^a + \frac{1}{2} y^\beta \delta_\beta^c \delta_\alpha^b f_{cb}{}^a + \ldots , \qquad e_\alpha^i(y) = \frac{1}{2} y^\beta \delta_\beta^c \delta_\alpha^b f_{cb}{}^i + \ldots . \tag{2.17}$$

From the equations above is evident that at $y = 0$, $e_\alpha^a = \delta_\alpha^a$ and $e_\alpha^i = 0$, a fact I shall use later.

I proceed by calculating the connection on S/R which is described by a connection-form $\theta^a{}_b$. In the general case where torsion may be non-zero, one calculate first the torsionless part $\omega^a{}_b$ by setting the torsion form T^a equal to zero,

$$T^a = de^a + \omega^a{}_b \wedge e^b = 0\,, \tag{2.18}$$

while using the Maurer-Cartan equation,

$$de^a = \frac{1}{2} f^a{}_{bc} e^b \wedge e^c + f^a{}_{bi} e^b \wedge e^i\,, \tag{2.19}$$

I see that the condition of having vanishing torsion is solved by

$$\omega^a{}_b = -f^a{}_{ib} e^i - D^a{}_{bc} e^c\,, \tag{2.20}$$

where

$$D^a{}_{bc} = \frac{1}{2} g^{ad} [f_{db}{}^e g_{ec} + f_{cb}{}^e g_{de} - f_{cd}{}^e g_{be}]\,.$$

The D's can be related to f's by a rescaling [18]

$$D^a{}_{bc} = (\lambda^a \lambda^b / \lambda^c) f^a{}_{bc}\,,$$

where the λ's depend on the coset radii. Note that in general the rescalings change the antisymmetry properties of f's, while in the case of equal radii $D^a{}_{bc} = \frac{1}{2} f^a{}_{bc}$. Note also that the connection-form $\omega^a{}_b$ is S-invariant. This means that parallel transport commutes with the S action [25]. Then the most general form of an S-invariant connection on S/R would be

$$\omega^a{}_b = f^a{}_{ib} e^i + J^a{}_{cb} e^c\,, \tag{2.21}$$

with J an R-invariant tensor, i.e.

$$\delta J_{cb}{}^a = -f_{ic}{}^d J_{db}{}^a + f_{id}{}^a J_{cb}{}^d - f_{ib}{}^d J_{cd}{}^a = 0\,.$$

This condition is satisfied by the D's as can be proven using the Jacobi identity.

In the case of non-vanishing torsion one has

$$T^a = de^a + \theta^a{}_b \wedge e^b\,, \tag{2.22}$$

where

$$\theta^a{}_b = \omega^a{}_b + \tau^a{}_b\,,$$

with

$$\tau^a{}_b = -\frac{1}{2} \Sigma^a{}_{bc} e^c\,, \tag{2.23}$$

while the contorsion $\Sigma^a{}_{bc}$ is given by

$$\Sigma^a{}_{bc} = T^a{}_{bc} + T_{bc}{}^a - T_{cb}{}^a \tag{2.24}$$

in terms of the torsion components $T^a{}_{bc}$. Therefore in general and for the case of non-symmetric cosets the connection-form $\theta^a{}_b$ is

$$\theta^a{}_b = -f^a{}_{ib}e^i - (D^a{}_{bc} + \frac{1}{2}\Sigma^a{}_{bc})e^c = -f^a{}_{ib}e^i - G^a{}_{bc}e^c \,. \tag{2.25}$$

The natural choice of torsion which would generalise the case of equal radii [36–40], $T^a{}_{bc} = \eta f^a{}_{bc}$ would be $T^a{}_{bc} = 2\tau D^a{}_{bc}$ except that the D's do not have the required symmetry properties. Therefore one has to define Σ as a combination of D's which makes Σ completely antisymmetric and S-invariant according to the definition given above, i.e.

$$\Sigma_{abc} \equiv 2\tau(D_{abc} + D_{bca} - D_{cba}) \,. \tag{2.26}$$

In this general case the Riemann curvature two-form is given by [18]

$$R^a{}_b = \left[-\frac{1}{2} f_{ib}{}^a f_{de}{}^i - \frac{1}{2} G_{cb}{}^a f_{de}{}^c + \frac{1}{2}(G_{dc}{}^a G_{eb}{}^c - G_{ec}{}^a G_{db}{}^c) \right] e^d \wedge e^e \,, \tag{2.27}$$

whereas the Ricci tensor $R_{ab} = R^d{}_{adb}$ is

$$R_{ab} = G_{ba}{}^c G_{dc}{}^d - G_{bc}{}^d G_{da}{}^c - G_{ca}{}^d f_{db}{}^c - f_{ia}{}^d f_{db}{}^i \,. \tag{2.28}$$

By choosing vanishing parameter τ in the eqs. (2.26) and (2.25) above one obtains the *Riemannian connection*, $\theta_R{}^a{}_b = -f^a{}_{ib}e^i - D^a{}_{bc}e^c$. On the other hand, by adjusting the radii and τ one can obtain the *canonical connection*, $\theta_C{}^a{}_b = -f^a{}_{bi}e^i$ which is an R-gauge field [36,37]. In general though the $\theta^a{}_b$ connection is an $SO(6)$ field, i.e. lives on the tangent space of the six-dimensional cosets I consider and describes their general holonomy. In subsection 2.2.3 I will show how the $G^{ab}{}_c$ term of eq. (2.25) it is connected with the geometrical and torsion contributions that the masses of the surviving four-dimensional gaugini acquire. Since I am interested here in four-dimensional models without light supersymmetric particles I keep $\theta^a{}_b$ general. Concerning the Ricci tensor, R_{ab} one can make appropriate adjustments of the torsion to set it equal to zero [36,37], thus defining a *Ricci flattening connection*.

2.2.2 Reduction of a D-dimensional Yang-Mills-Dirac Lagrangian

The group S acts as a symmetry group on the extra coordinates. The CSDR scheme demands that an S-transformation of the extra d coordinates is a gauge transformation of the fields that are defined on $M^4 \times (S/R)$, thus a gauge invariant Lagrangian written on this space is independent of the extra coordinates.

To see this in detail let me consider a D-dimensional Yang-Mills-Dirac theory with gauge group G defined on a manifold M^D which as stated will be compactified to $M^4 \times (S/R)$, $D = 4 + d$, $d = \dim(S) - \dim(R)$

$$A = \int d^4x d^dy \sqrt{-g} \Big[-\frac{1}{4} Tr\,(F_{MN}F_{K\Lambda})\, g^{MK} g^{N\Lambda} + \frac{i}{2}\overline{\psi}\Gamma^M D_M \psi \Big] \,, \tag{2.29}$$

where

$$D_M = \partial_M - \theta_M - A_M \,, \tag{2.30}$$

with

$$\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{N\Lambda} \tag{2.31}$$

the spin connection of M^D, and

$$F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N] \,, \tag{2.32}$$

where M, N run over the D-dimensional space. The fields A_M and ψ are, as explained, symmetric in the sense that any transformation under symmetries of S/R is compensated by gauge transformations. The fermion fields can be in any rep. F of G unless a further symmetry is required. Here, since I assume dimensional reductions of $\mathcal{N} = 1$ supersymmetric gauge theory, the higher dimensional fermions have to transform in the adjoint of the higher dimensional gauge group. To be more specific let ξ_A^α, $A = 1, \ldots, \dim(S)$, be the Killing vectors which generate the symmetries of S/R and W_A the compensating gauge transformation associated with ξ_A. Define next the infinitesimal coordinate transformation as $\delta_A \equiv L_{\xi_A}$, the Lie derivative with respect to ξ, then one has for the scalar, vector and spinor fields,

$$\begin{aligned}
&\delta_A \phi = \xi_A^\alpha \partial_\alpha \phi = D(W_A)\phi \,, \\
&\delta_A A_\alpha = \xi_A^\beta \partial_\beta A_\alpha + \partial_\alpha \xi_A^\beta A_\beta = \partial_\alpha W_A - [W_A, A_\alpha] \,, \\
&\delta_A \psi = \xi_A^\alpha \psi - \frac{1}{2} G_{Abc}\Sigma^{bc}\psi = D(W_A)\psi \,.
\end{aligned} \tag{2.33}$$

W_A depend only on internal coordinates y and $D(W_A)$ represents a gauge transformation in the appropriate rep. of the fields. G_{Abc} represents a tangent space rotation of the spinor fields. The variations δ_A satisfy, $[\delta_A, \delta_B] = f_{AB}{}^C \delta_C$ and lead to the following consistency relation for W_A's,

$$\xi_A^\alpha \partial_\alpha W_B - \xi_B^\alpha \partial_\alpha W_A - [W_A, W_B] = f_{AB}{}^C W_C \,. \tag{2.34}$$

Let me know examine how the W_A will change when the fields are transformed under the gauge group. Consider the gauge transformation of the scalar field ϕ, $\phi \to \phi^{(g)} = D(g)\phi$. Then the W_A needed to compensate an S-transformation acting on $\phi^{(g)}$ will be

$$W_A^{(g)} = g\, W_A\, g^{-1} + (\delta_A g) g^{-1} \,. \tag{2.35}$$

The requirement that W_A transforms according to eq. (2.35) under gauge transformation ensure that the constraints (2.33) remain invariant under general coordinate and gauge transformations.

In order to solve the constraints I make use of the transitivity of the action of S on S/R. Then the value of a symmetric field at any point on S/R is determined by its value at

the origin and an S-transformation. Therefore a convenient point to do the calculations is the origin $y = 0$ while using the gauge freedom (2.35) one can make the choice

$$W_a(y^\alpha = 0) = 0\,. \tag{2.36}$$

Under these assumptions eq. (2.34) yields

$$\partial_a W_b - \partial_b W_a = f_{ab}{}^i W_i\,, \tag{2.37}$$
$$\partial_a W_i = 0\,, \tag{2.38}$$
$$[W_i, W_j] = -f_{ij}{}^k W_k\,. \tag{2.39}$$

From eq. (2.38) I see that the W_i are constants over the coset. Defining

$$J_i \equiv -W_i\,, \tag{2.40}$$

eq. (2.40) implies that the J_i form the algebra of R. Since the W live by definition in the Lie algebra of the gauge group G, eq. (2.39) makes sense only if R is embedded in G. In that case the J_i are the generators of an R-subgroup, R_G of G. The eq. (2.37) will be useful in the calculation of the potential.

I proceed by analysing the constraints on the fields in the theory. A gauge field A_M on M^D splits into A_μ on M^4 and A_a on S/R; A_μ behaves as a scalar under S-transformations and lies in the adjoint rep. of G. From the first of eqs (2.33) one obtains at $y = 0$

$$\partial_a A_\mu = 0\,, \qquad [J_i, A_\mu] = 0\,. \tag{2.41}$$

The first of the above equations indicates that the four-dimensional gauge field is completely independent of the coset space coordinates. Furthermore the gauge group in four dimensions is dictated by the second of eqs (2.41). Since A_μ is commutes with J_i, which are the generators of R_G in G, the surviving gauge symmetry H is that subgroup of G which commutes with R. In other words is the centraliser of R in G, i.e., $H = C_G(R_G)$.

The remaining components of the higher dimensional gauge field A_α become vectors under the coset space transformations. As they will be the scalar fields in the resulting four-dimensional theory one can write $\Phi_a = e^\alpha_a A_\alpha$. The second of the eqs (2.33) at $y = 0$ implies

$$\partial_a \Phi_b - \partial_b W_a = \frac{1}{2} f_{ab}{}^c \Phi_c\,, \tag{2.42a}$$
$$[J_i, \Phi_a] = \phi_{ia}{}^c \Phi_c\,. \tag{2.42b}$$

Eq. (2.42a) will be useful when I will calculate the potential of the theory. From eq (2.42b) I see that the Φ_a act as interwining operator connecting the induced reps of R in G and in S. Indeed, I have already shown that the J_i form an R-subalgebra of G. Denoting by G_s the generators of the gauge group G and its structure constants by g_{str}, one can write $\Phi_a = \Phi^s_a G_s$ and eq. (2.42b) takes the form

$$\Phi^s_a g_{ist} = f_{ia}{}^c \Phi^t_c\,, \tag{2.43}$$

or

$$\Phi_a^s (M_i)_{st} = (M_i')_{ac} \Phi_c^t \,, \tag{2.44}$$

where $(M_i)_{st} = -g_{ist}$ and $(M_i')_{ac} = -f_{ia}{}^c$.

In general M_i and M_i' are reducible representations of R. With a suitable choice of basis in each case one can write M_i and M_i' in a block diagonal form. Then each submatrix form an irreducible representation (irrep.) of R. Let me consider the submatrices M_p and M_q' corresponding to two irreps of R. Then restricting eq. (2.44) to these particular submatrices I obtain

$$\Phi_a^{s(qp)} (M_p)_{st} = (M_q')_{ac} \Phi_c^{\prime t(qp)} \,. \tag{2.45}$$

Since M_p and M_q' are of irreps, $\Phi^{(qp)}$ must have linearly independent rows and columns. Then if M_p and M_q' are of different dimension, then obviously the $\Phi^{(qp)}$ has to vanish. Furthermore, if M_p and M_q' have the same dimension but are different irreps, $\Phi^{(qp)}$ vanishes again because otherwise eq. (2.45) implies that M_p and M_q' are related by a change of basis. Finally, if M_p and M_q' are the same rep., then eq. (2.45) states that $\Phi^{(qp)}$ commutes with all the matrices in that rep. and by Schur's lemma it must be multiple of the identity matrix, i.e., $\Phi^{(qp)} = \phi^{(qp)}(x)\mathbb{1}$. This shows that in order to find the rep. of the gauge group H under which the ϕ transform in four dimensions, one has to decompose S under R

$$\begin{aligned} S &\supset R \\ \operatorname{adj} S &= \operatorname{adj} R + \mathrm{v} \end{aligned} \tag{2.46}$$

and the gauge group G according to the embedding

$$\begin{aligned} G &\supset R_G \times H \\ \operatorname{adj} G &= (\operatorname{adj} R, 1) + (1, \operatorname{adj} H) + \sum (r_i, h_i) \,. \end{aligned} \tag{2.47}$$

Then if $\mathrm{v} = \sum s_i$, where each s_i is an irrep. of R, there survives an h_i multiplet for every pair (r_i, s_i), where r_i and s_i are identical irreps of R.

Turning next to the fermion fields [13, 18, 27, 41–45] similarly to scalars, they act as intertwining operators between induced reps acting on G and the tangent space of S/R, $SO(d)$. Proceeding along similar lines as in the case of scalars to obtain the rep. of H under which the four-dimensional fermions transform, I have to decompose the rep. F of the initial gauge group in which the fermions are assigned under $R_G \times H$, i.e.

$$F = \sum (t_i, h_i) \,, \tag{2.48}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j \,. \tag{2.49}$$

Then for each pair t_i and σ_i, where t_i and σ_i are identical irreps there is an h_i multiplet of spinor fields in the four-dimensional theory. In order however to obtain chiral fermions in the effective theory one has to impose further requirements. I first impose the Weyl condition in D dimensions. In $D = 4n + 2$ dimensions which is the case at hand, the decomposition of the left handed, say spinor under $SU(2) \times SU(2) \times SO(d)$ is

$$\sigma_D = (2, 1, \sigma_d) + (1, 2, \overline{\sigma}_d)\,. \tag{2.50}$$

Furthermore in order to be $\sigma_d \neq \overline{\sigma}_d$ the coset space S/R must be such that rank$(R) =$ rank(S) [18,46]. The six-dimensional coset spaces which satisfy this condition are listed in tables 2.1 and 2.2. Then under the $SO(d) \supset R$ decomposition one has

$$\sigma_d = \sum \sigma_k,\ \ \overline{\sigma}_d = \sum \overline{\sigma}_k\,. \tag{2.51}$$

In the following chapters I assume that the higher dimensional theory is $\mathcal{N} = 1$ supersymmetric. Therefore the higher dimensional fermionic fields have to be considered transforming in the adjoint of E_8 which is vectorlike. In this case each term (t_i, h_i) in eq. (2.48) will be either self-conjugate or it will have a partner $(\overline{t}_i, \overline{h}_i)$. According to the rule described in eqs. (2.48), (2.49) and considering σ_d I will have in four dimensions left-handed fermions transforming as $f_L = \sum h_k^L$. It is important to notice that since σ_d is non self-conjugate, f_L is non self-conjugate too. Similarly from $\overline{\sigma}_d$ I will obtain the right-handed rep. $f_R = \sum \overline{h}_k^R$ but as I have assumed that F is vector-like, $\overline{h}_k^R \sim h_k^L$. Therefore there will appear two sets of Weyl fermions with the same quantum numbers under H. This is already a chiral theory but still one can go further and try to impose the Majorana condition in order to eliminate the doubling of the fermion spectrum. However this is not required in the present case of interest where I apply the Hosotani mechanism for the further breaking of the gauge symmetry, as I will explain in chapter 3.

An important requirement is that the resulting four-dimensional particle physics models should be anomaly free. Starting with an anomaly free theory in higher dimensions, Witten [47] has given the condition to be fulfilled in order to obtain anomaly free four-dimensional theories. The condition restricts the allowed embeddings of R into G by relating them with the embedding of R into $SO(6)$ [18, 48]. To be more specific if Λ_a are the generators of R into G and T_a are the generators of R into $SO(6)$ the condition reads

$$Tr(\Lambda_a \Lambda_b) = 30\, Tr(T_a T_b)\,. \tag{2.52}$$

According to ref. [48] the anomaly cancellation condition (2.52) is automatically satisfied for the choice of embedding

$$E_8 \supset SO(6) \supset R\,, \tag{2.53}$$

which I adopt here. Furthermore, concerning the abelian group factors of the four-dimensional gauge theory, note that the corresponding gauge bosons surviving in four dimensions become massive at the compactification scale [47, 49] and therefore, they do not contribute in the anomalies; they correspond only to global symmetries.

Table 2.1: Six-dimensional symmetric cosets spaces with rank(R) = rank(S)**.** *The freely acting discrete symmetries* Z(S) *and* W *for each case are listed. The transformation properties under* R *are also noted.*

Case	$6D$ Coset Spaces	$\mathbf{Z}(S)$	W	V	F
a	$\frac{SO(7)}{SO(6)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{6}\leftrightarrow\mathbf{6}$	$\mathbf{4}\leftrightarrow\overline{\mathbf{4}}$
b	$\frac{SU(4)}{SU(3)\times U(1)}$	$\mathbb{Z}_4$	$\mathbb{1}$	$\mathbf{6}=\mathbf{3}_{(-2)}+\overline{\mathbf{3}}_{(2)}$ $-$	$\mathbf{4}=\mathbf{1}_{(3)}+\mathbf{3}_{(-1)}$ $-$
c	$\frac{Sp(4)}{(SU(2)\times U(1))_{max}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{6}=\mathbf{3}_{(-2)}+\mathbf{3}_{(2)}$ $\mathbf{3}_{(-2)}\leftrightarrow\mathbf{3}_{(2)}$	$\mathbf{4}=\mathbf{1}_{(3)}+\mathbf{3}_{(-1)}$ $\mathbf{1}_{(3)}\leftrightarrow\mathbf{1}_{(-3)}$ $\mathbf{3}_{(-1)}\leftrightarrow\mathbf{3}_{(1)}$
d	$\left(\frac{SU(3)}{SU(2)\times U(1)}\right)\times\left(\frac{SU(2)}{U(1)}\right)$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbf{6}=\mathbf{1}_{(0,2a)}+\mathbf{1}_{(0,-2a)}$ $+\mathbf{2}_{(b,0)}+\mathbf{2}_{(-b,0)}$ $\mathbf{1}_{(0,2a)}\leftrightarrow\mathbf{1}_{(0,-2a)}$	$\mathbf{4}=\mathbf{2}_{(0,a)}+\mathbf{1}_{(b,-a)}+\mathbf{1}_{(-b,-a)}$ $\mathbf{2}_{(0,a)}\leftrightarrow\mathbf{2}_{(0,-a)}$ $\mathbf{1}_{(b,-a)}\leftrightarrow\mathbf{1}_{(b,a)}$ $\mathbf{1}_{(-b,-a)}\leftrightarrow\mathbf{1}_{(-b,a)}$
e	$\left(\frac{Sp(4)}{SU(2)\times SU(2)}\right)\times\left(\frac{SU(2)}{U(1)}\right)$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2$	$\mathbf{6}=(\mathbf{2},\mathbf{2})_{(0)}+(\mathbf{1},\mathbf{1})_{(2)}+(\mathbf{1},\mathbf{1})_{(-2)}$ ($\mathbb{Z}_2$ of $SU(2)/U(1)$) $(\mathbf{1},\mathbf{1})_{(2)}\leftrightarrow(\mathbf{1},\mathbf{1})_{(-2)}$	$\mathbf{4}=(\mathbf{2},\mathbf{1})_{(1)}+(\mathbf{1},\mathbf{2})_{(-1)}$ $(\mathbf{2},\mathbf{1})_{(1)}\leftrightarrow(\mathbf{2},\mathbf{1})_{(-1)}$ $(\mathbf{1},\mathbf{2})_{(1)}\leftrightarrow(\mathbf{1},\mathbf{2})_{(-1)}$
f	$\left(\frac{SU(2)}{U(1)}\right)^3$	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^3$	$\mathbf{6}=(2a,0,0)+(0,2b,0)+(0,0,2c)$ $+(-2a,0,0)+(0,-2b,0)+(0,0,-2c)$ each $\mathbb{Z}_2$ changes the sign of a,b, c	$\mathbf{4}=(a,b,c)+(-a,-b,c)$ $+(-a,b,-c)+(a,-b,-c)$ each $\mathbb{Z}_2$ changes the sign of a,b, c

Table 2.2: Six-dimensional non-symmetric cosets spaces with rank(R) = rank(S)**.** *The available freely acting discrete symmetries* Z(S) *and* W *for each case are listed. The transformation properties under* R *are also noted.*

Case	$6D$ Coset Spaces	$\mathbf{Z}(S)$	W	V	F
a′	$\frac{G_2}{SU(3)}$	**1**	$\mathbb{Z}_2$	$\mathbf{6}=\mathbf{3}+\overline{\mathbf{3}}$ $\mathbf{3}\leftrightarrow\overline{\mathbf{3}}$	$\mathbf{4}=\mathbf{1}+\mathbf{3}$ $\mathbf{1}\leftrightarrow\mathbf{1}$ $\mathbf{3}\leftrightarrow\overline{\mathbf{3}}$
b′	$\frac{Sp(4)}{(SU(2)\times U(1))_{nonmax}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{6}=\mathbf{1}_{(2)}+\mathbf{1}_{(-2)}+\mathbf{2}_{(1)}+\mathbf{2}_{(-1)}$ $\mathbf{1}_{(2)}\leftrightarrow\mathbf{1}_{(-2)}$ $\mathbf{2}_{(1)}\leftrightarrow\mathbf{2}_{(-1)}$	$\mathbf{4}=\mathbf{1}_{(0)}+\mathbf{1}_{(2)}+\mathbf{2}_{(-1)}$ $\mathbf{1}_{(2)}\leftrightarrow\mathbf{1}_{(-2)}$ $\mathbf{1}_{(0)}\leftrightarrow\mathbf{1}_{(0)}$ $\mathbf{2}_{(1)}\leftrightarrow\mathbf{2}_{(-1)}$
c′	$\frac{SU(3)}{U(1)\times U(1)}$	$\mathbb{Z}_3$	$\mathbf{S}_3$	$\mathbf{6}=(a,c)+(b,d)+(a+b,c+d)$ $+(-a,-c)+(-b,-d)$ $+(-a-b,-c-d)$	$\mathbf{4}=(0,0)$ $+(a,c)+(b,d)+(-a-b,-c-d)$
			$\mathbb{Z}_2$	$(b,d)\leftrightarrow(-b,-d)$ $(a+b,c+d)\leftrightarrow(a,c)$ $(-a,-c)\leftrightarrow(-a-b,-c-d)$	$(b,d)\leftrightarrow(-b,-d)$ $(a,c)\leftrightarrow(a+b,c+d)$ $(-a-b,-c-d)\leftrightarrow(-a,-c)$
			$\mathbb{Z}_2$	$(b,d)\leftrightarrow(a+b,c+d)$ $(a,c)\leftrightarrow(-a,-c)$ $(-b,-d)\leftrightarrow(-a-b,-c-d)$	$(b,d)\leftrightarrow(a+b,c+d)$ $(a,c)\leftrightarrow(-a,-c)$ $(-a-c,-b-d)\leftrightarrow(-b,-d)$
			$\mathbb{Z}_2$	$(b,d)\leftrightarrow(-a,-c)$ $(a+b,c+d)\leftrightarrow(-a-b,-c-d)$ $(a,c)\leftrightarrow(-b,-d)$	$(b,d)\leftrightarrow(-a,-c)$ $(a,c)\leftrightarrow(-b,-d)$ $(-a-b,-c-d)\leftrightarrow(a+b,c+d)$

2.2.3 The four-dimensional theory

Next let me obtain the four-dimensional effective action. Assuming that the metric is block diagonal, taking into account all the constraints and integrating out the extra coordinates I obtain in four dimensions the following Lagrangian

$$A = C \int d^4x \left(-\frac{1}{4} F^t_{\mu\nu} F^{t\mu\nu} + \frac{1}{2} (D_\mu \phi_a)^t (D^\mu \phi^a)^t + V(\phi) + \frac{i}{2} \overline{\psi} \Gamma^\mu D_\mu \psi + \frac{i}{2} \overline{\psi} \Gamma^a D_a \psi \right), \tag{2.54}$$

where $D_\mu = \partial_\mu - A_\mu$ and $D_a = \partial_a - \theta_a - \phi_a$ with $\theta_a = \frac{1}{2}\theta_{abc}\Sigma^{bc}$ the connection of the coset space and Σ^{bc} the $SO(6)$ generators. With C I denote the volume of the coset space. The potential $V(\phi)$ is given by

$$V(\phi) = -\frac{1}{4} g^{ac} g^{bd} Tr(f_{ab}{}^C \phi_C - [\phi_a, \phi_b])(f_{cd}{}^D \phi_D - [\phi_c, \phi_d]) \,, \tag{2.55}$$

where, $A = 1, \ldots, \dim(S)$ and f ' s are the structure constants appearing in the commutators of the generators of the Lie algebra of S. The expression (2.55) for $V(\phi)$ is only formal because ϕ_a must satisfy the constraints coming from eq. (2.33),

$$f_{ai}{}^D \phi_D - [\phi_a, \phi_i] = 0 \,, \tag{2.56}$$

where the ϕ_i generate R_G. These constraints imply that some components ϕ_a's are zero, some are constants and the rest can be identified with the genuine Higgs fields according to the rules presented in eqs (2.46) and (2.47).

When $V(\phi)$ is expressed in terms of the unconstrained independent Higgs fields, it remains a quartic polynomial which is invariant under gauge transformations of the final gauge group H, and its minimum determines the vacuum expectation values of the Higgs fields [50–53]. The minimisation of the potential is in general a difficult problem. If however S has an isomorphic image S_G in G which contains R_G in a consistent way then it is possible to allow the ϕ_a to become generators of S_G. That is $\overline{\phi}_a =< \phi^i > Q_{ai} = Q_a$ with $< \phi^i > Q_{ai}$ suitable combinations of G generators, Q_a a generator of S_G and a is also a coset-space index. Then

$$\begin{aligned} \overline{F}_{ab} &= f_{ab}{}^i Q_i + f_{ab}{}^c \overline{\phi}_c - [\overline{\phi}_a, \overline{\phi}_b] \\ &= f_{ab}{}^i Q_i + f_{ab}{}^c Q_c - [Q_a, Q_b] = 0 \end{aligned}$$

because of the commutation relations of S. Thus I have proven that $V(\phi = \overline{\phi}) = 0$ which furthermore is the minimum, because V is positive definite. Furthermore, the four-dimensional gauge group H breaks further by these non-zero vacuum expectation values of the Higgs fields to the centraliser K of the image of S in G, i.e. $K = C_G(S)$ [18,50–53]. This can been seen if I examine a gauge transformation of ϕ_a by an element h of H. Then I have

$$\phi_a \to h \phi_a h^{-1}, \qquad h \in H$$

We note that the v.e.v. of the Higgs fields is gauge invariant for the set of h's that commute with S. That is h belongs to a subgroup K of H which is the centraliser of S_G in G. It should be stressed that the four-dimensional fermions of this class of models acquire large masses due to geometrical contributions at the compactification scale [18, 45]. In general it can be proven [18] that dimensional reduction over a symmetric coset space always gives a potential of spontaneous breaking form which is not the case of non-symmetric cosets of more than one radii.

In the fermion part of the Lagrangian the first term is just the kinetic term of fermions, while the second is the Yukawa term [45, 54]. The last term in (2.54) can be written as

$$L_D = \frac{i}{2}\overline{\psi}\Gamma^a(\partial_a - \frac{1}{2}f_{ibc}e^i_\gamma e^\gamma_a \Sigma^{bc} - \frac{1}{2}G_{abc}\Sigma^{bc} - \phi_a)\psi = \frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi + \overline{\psi}V\psi\,, \tag{2.57}$$

where

$$\nabla_a = \partial_a - \frac{1}{2}f_{ibc}e^i_\gamma e^\gamma_a\Sigma^{bc} - \phi_a, \tag{2.58}$$

$$V = -\frac{i}{4}\Gamma^a G_{abc}\Sigma^{bc}\,, \tag{2.59}$$

and G_{abc} is given in eq. (2.25) as $G^a{}_{bc} = D^a{}_{bc} + \frac{1}{2}\Sigma^{bc}$. I have already noticed that according to the CSDR constraints, $\partial_a\psi = 0$. Furthermore one can consider the Lagrangian at the point $y = 0$, due to its invariance under S-transformations, and according to the discussion in subsection 2.2.1 $e^i_\gamma = 0$ at that point. Therefore (2.58) becomes just $\nabla_a = \phi_a$ and the term $\frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi$ in eq. (2.57) is exactly the Yukawa term.

Let me examine now the last term appearing in (2.57). One can show easily that the operator V anticommutes with the six-dimensional helicity operator [18]. Furthermore one can show that V commutes with the $T_i = -\frac{1}{2}f_{ibc}\Sigma^{bc}$ [T_i close the R-subalgebra of $SO(6)$]. In turn I can draw the conclusion, exploiting Schur's lemma, that the non-vanishing elements of V are only those which appear in the decomposition of both $SO(6)$ irreps 4 and $\overline{4}$, e.g. the singlets. Since this term is of pure geometric nature, I reach the conclusion that the singlets in 4 and $\overline{4}$ will acquire large geometrical masses, a fact that has serious phenomenological implications. First note this is characteristic of the non-symmetric cosets only. In [29, 30] was found that dimensional reduction of supersymmetric theories defined in higher dimensions over non-symmetric coset spaces results in particle physics models with a softly broken supersymmetry. The surviving four-dimensional fermions coming from the identification of singlets of $SO(6)$ irreps **4** live necessarily in the adjoint of H as was stated in subsection 2.2.2 thus being the gaugini of the model. Then, according to our previous argument, will receive masses comparable to the compactification scale. In the case of the symmetric cosets though, the V operator is absent since $f_{ab}{}^c$ are vanishing by definition.

2.3 Remarks on Grand Unified theories resulting from CSDR

Here I make few remarks on models resulting from the coset space dimensional reduction of an $\mathcal{N} = 1$, E_8 gauge theory which is defined on a ten-dimensional compactified space $M^D = M^4 \times (S/R)$. The coset spaces S/R I consider are listed in the first column of tables 2.1 and 2.2. In order to obtain four-dimensional GUTs potentially with phenomenological interest, namely $\mathcal{H} = E_6$, $SO(10)$ and $SU(5)$, is sufficient to consider only embeddings of the isotropy group R of the coset space in

$$\mathcal{R} = C_{E_8}(\mathcal{H}) = SU(3)\,, \qquad \text{for} \quad \mathcal{H} = E_6\,, \tag{2.60a}$$
$$\mathcal{R} = C_{E_8}(\mathcal{H}) = SO(6) \sim SU(4)\,, \qquad \text{for} \quad \mathcal{H} = SO(10)\,, \tag{2.60b}$$
$$\mathcal{R} = C_{E_8}(\mathcal{H}) = SU(5)\,, \qquad \text{for} \quad \mathcal{H} = SU(5)\,. \tag{2.60c}$$

As it was noted in subsection 2.2.2 the anomaly cancellation condition (2.52) is satisfied automatically for the choice of embedding

$$E_8 \supset SO(6) \supset R\,, \tag{2.61}$$

which I adopt here. This requirement is trivially fullfiled for the case of $R \hookrightarrow \mathcal{R}$ embeddings of eq. (2.60b) which lead to $SO(10)$ GUTs in four dimensions. It is obviously also satisfied for the case of $R \hookrightarrow \mathcal{R}$ embeddings of eq. (2.60a) since $SU(3) \subset SO(6)$. The above case leads to E_6 GUTs in four dimensions. Finally, $R \hookrightarrow \mathcal{R}$ embeddings of eq. (2.60c) are excluded since the requirement (2.61) cannot be satisfied.

2.4 Reduction of $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory over $SO(7)/SO(6)$

Let a $\mathcal{N} = 1$, $G = E_8$ Yang-Mills-Dirac theory defined in ten dimensions which are compactified in the form $M^4 \times S^6$, $S^6 \sim SO(7)/SO(6)$. Furthermore let me consider Weyl fermions belonging in the adjoint of $G = E_8$ and the embedding of $R = SO(6)$ into E_8 suggested by the decomposition

$$\begin{aligned} E_8 \;&\supset SO(16) \supset SO(6) \times SO(10) \\ \mathbf{248} &= (\mathbf{1}, \mathbf{45}) + (\mathbf{6}, \mathbf{10}) + (\mathbf{15}, \mathbf{1}) + (\mathbf{4}, \mathbf{16}) + (\overline{\mathbf{4}}, \overline{\mathbf{16}})\,. \end{aligned} \tag{2.62}$$

If only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SO(6)) = SO(10)\,.$$

According to table 2.1, the $R = SO(6)$ content of vector and spinor of $B_0 = S/R = SO(7)/SO(6)$ is $\mathbf{6}$ and $\mathbf{4}$, respectively. Then applying the CSDR rules (2.46), (2.47)

and (2.48), (2.49) the four-dimensional theory would contain scalars transforming as **10** under the $H = SO(10)$ gauge group and two copies of chiral fermions belonging in the $\mathbf{16}_L$ of H.

According to subsection 2.2.3, dimensional reduction over the $SO(7)/SO(6)$ symmetric coset space leads to a four-dimensional action with a potential of spontaneously symmetry breaking form. In addition, note that the isometry group of the coset, $SO(7)$, is embeddable in E_8 as

$$\begin{array}{ccc} E_8 \supset SO(7) \times & SO(9) \\ \cup & \cap \\ SO(6) \times & SO(10) \,. \end{array}$$

Then, according to the theorem mentioned in subsection 2.2.3, the final gauge group is

$$\mathcal{H} = C_{E_8}(SO(7)) = SO(9)\,,$$

i.e. the **10** of $SO(10)$ obtains a v.e.v. leading to the spontaneous symmetry breaking

$$\begin{aligned} &SO(10) \rightarrow SO(9) \\ &\mathbf{10} = \langle \mathbf{1} \rangle + \mathbf{9}\,. \end{aligned} \tag{2.63}$$

Chapter 3

Wilson flux breaking mechanism in CSDR

In the previous section an example of the CSDR mechanism was given. I assumed an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory defined on the ten-dimensional space $M^4 \times S/R$, $S/R = SO(7)/SO(6)$. The resulting four-dimensional gauge theory was an $SO(10)$ GUT with scalars transforming as **10** of $SO(10)$ and two copies of chiral fermion transforming as $\mathbf{16}_L$ left-handed multiplets. The surviving scalars of the theory being in the fundamental representation of the gauge group are not able to provide the appropriate superstrong symmetry breaking towards the SM. This is an intrinsic characteristic of the CSDR scheme. As described in subsection 2.2.2 the surviving scalars in four dimensions are calculated by comparing the R content of the vector $SO(6)$ under the $SO(6) \supset R$ decomposition with the irreps occurring in the vector of the $G \supset R_G \times H$ embedding. Therefore it is impossible to transform in the adjoint of the surviving gauge group as an appropriate superstrong symmetry breaking of GUTs requires. As a way out it has been suggested [20] to take advantage of non-trivial topological properties of the compactification coset space, apply the Hosotani or Wilson flux breaking mechanism [21–23] and break the gauge symmetry of the theory further. Application of this mechanism imposes further constraints in the scheme.

Here I first recall the Wilson flux breaking mechanism, I make some remarks on specific cases which potentially lead to interesting models and I finally calculate the actual symmetry breaking patterns of the GUTs.

3.1 Wilson flux breaking mechanism

Let me briefly recall the Wilson flux mechanism for breaking spontaneously a gauge theory. Then instead of considering a gauge theory on $M^4 \times B_0$, with B_0 a simply connected manifold, and in my case a coset space $B_0 = S/R$, I consider a gauge theory on $M^4 \times B$, with $B = B_0/F^{S/R}$ and $F^{S/R}$ a freely acting discrete symmetry* of B_0. It turns out that B becomes multiply connected, which means that there will be contours not contractible to a point due to holes in the manifold. For each element $g \in F^{S/R}$, I pick up an element U_g in H, i.e. in the four-dimensional gauge group of the reduced theory, which can be represented as the Wilson loop

$$U_g = \mathcal{P}exp\left(-i \int_{\gamma_g} T^a A^a_M(x) dx^M\right) , \tag{3.1}$$

where $A^a_M(x)$ are vacuum H fields with group generators T^a, γ_g is a contour representing the abstract element g of $F^{S/R}$, and $\mathcal{P}$ denotes the path ordering.

Now if γ_g is chosen not to be contractible to a point, then $U_g \neq 1$ although the vacuum field strength vanishes everywhere. In this way an homomorphism of $F^{S/R}$ into H is induced with image T^H, which is the subgroup of H generated by $\{U_g\}$. A field $f(x)$ on B_0 is obviously equivalent to another field on B_0 which obeys $f(g(x)) = f(x)$ for every $g \in F^{S/R}$. However in the presence of the gauge group H this statement can be generalised to

$$f(g(x)) = U_g f(x) . \tag{3.2}$$

Next, one would like to see which gauge symmetry is preserved by the vacuum. The vacuum has $A^a_\mu = 0$ and I represent a gauge transformation by a space-dependent matrix $V(x)$ of H. In order to keep $A^a_\mu = 0$ and leave the vacuum invariant, $V(x)$ must be constant. On the other hand, $f \to Vf$ is consistent with equation (3.2), only if $[V, U_g] = 0$ for all $g \in F^{S/R}$. Therefore the H breaks towards the centraliser of T^H in H, $K' = C_H(T^H)$. In addition the matter fields have to be invariant under the diagonal sum

$$F^{S/R} \oplus T^H , \tag{3.3}$$

in order to satisfy eq. (3.2) and therefore survive in the four-dimensional theory.

*By freely acting I mean that for every element $g \in F$, except the identity, there exists no points of B_0 that remain invariant.

3.2 Further remarks concerning the use of the $F^{S/R}$

The discrete symmetries $F^{S/R}$, which act freely on coset spaces $B_0 = S/R$ are the center of S, $\mathrm{Z}(S)$ and the $\mathrm{W} = \mathrm{W}_S/\mathrm{W}_R$, with W_S and $mathrmW_R$ being the Weyl groups of S and R, respectively [18, 24, 55, 56]. The freely acting discrete symmetries, $F^{S/R}$, of the specific six-dimensional coset spaces under discussion are listed in the second and third column of tables 2.1 and 2.2. The $F^{S/R}$ transformation properties of the vector and spinor irreps under R are noted in the last two columns of the same tables.

According to the discussion in section 2.3, dimensional reduction over the six-dimensional coset spaces listed in tables 2.1 and 2.1, leads to E_6 and $SO(10)$ GUTs. My approach is to embed the $F^{S/R}$ discrete symmetries into four-dimensional $H = E_6$ and $SO(10)$ gauge groups. I make this choice only for bookkeeping reasons since, according to section 3.1, the actual topological symmetry breaking takes place in higher dimensions. Few remarks are in order. In both classes of models, namely E_6 and $SO(10)$ GUTs, the use of the discrete symmetry of the center of S, $\mathrm{Z}(S)$, cannot lead to phenomenologically interesting cases since various components of the irreps of the four-dimensional GUTs containing the SM fermions do not survive. The reason is that the irreps of H remain invariant under the action of the discrete symmetry, $\mathrm{Z}(S)$, and as a result the phase factors gained by the action of T^H cannot be compensated. Therefore the complete SM fermion spectrum cannot be invariant under $F^{S/R} \oplus T^H$ and survive. On the other hand, the use of the Weyl discrete symmetry can lead to better results. Models with potentially interesting fermion spectrum can be obtained employing at least one $\mathbb{Z}_2 \subset \mathrm{W}$. Then, the fermion content of the four-dimensional theory is found to transform in linear combinations of the two copies of the CSDR-surviving left-handed fermions. Details will be given in chapter 4. As I will discuss there, employing $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W} \times \mathrm{Z}(S)$ can also lead to interesting models.

Therefore the interesting cases for further study are

$$F^{S/R} = \begin{cases} \mathbb{Z}_2 \subseteq \mathrm{W} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \mathrm{W} \times \mathrm{Z}(S) \,. \end{cases} \tag{3.4}$$

3.3 Symmetry breaking patterns of E_6-like GUTs

Here I determine the image, T^H, that each of the discrete symmetries of eq. (3.4) induces in the gauge group $H = E_6$. I consider embeddings of the $F^{S/R}$ discrete symmetries into abelian subgroups of E_6 and examine their topologically induced symmetry breaking patterns [23]. These are realised by a diagonal matrix U_g of unit determinant, which as explained in section 3.1, has to be homomorphic to the considered discrete symmetry. In fig. 3.1 I present those E_6 decompositions which potentially lead to the SM gauge group structure [57].

$$
\begin{tikzcd}
 & F_4 \arrow[r] & SU(3)\times SU^c(3)\ (\mathbf{1}) & \\
E_6 \arrow[ur] \arrow[r] \arrow[dr] & SO(10) \arrow[r] & SU(5) \arrow[r] & SU^w(2)\times SU^c(3)\ (\mathbf{2}) \\
 & SU^w(3)\times SU(3)\times SU^c(3)\ (\mathbf{3}) & &
\end{tikzcd}
$$

Figure 3.1: *E_6 decompositions leading potentially to SM gauge group structure.*

3.3.1 The $\mathbb{Z}_2$ case

Embedding (1): $\mathbb{Z}_2 \hookrightarrow SU(3)$ **of** $E_6 \supset F_4 \supset SU(3) \times SU^c(3)$ **.**

Let me consider the maximal subgroups of E_6 and the corresponding decomposition of fundamental and adjoint irreps

$$
\begin{aligned}
& E_6 \supset F_4 \supset SU(3) \times SU^c(3) \\
& \mathbf{27}{=}(\mathbf{1},\mathbf{1}) + (\mathbf{8},\mathbf{1}) + (\mathbf{3},\mathbf{3}) + (\overline{\mathbf{3}},\overline{\mathbf{3}})\,, \\
& \mathbf{78}{=}(\mathbf{8},\mathbf{1}) + (\mathbf{3},\mathbf{3}) + (\overline{\mathbf{3}},\overline{\mathbf{3}}) + (\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{8}) + (\mathbf{6},\overline{\mathbf{3}}) + (\overline{\mathbf{6}},\mathbf{3})\,.
\end{aligned}
\tag{3.5}
$$

Let me also embed the $F^{S/R} = \mathbb{Z}_2$ discrete symmetry in the $SU(3)$ group factor above. There exist two distinct possibilities of embedding, either $\mathbb{Z}_2 \hookrightarrow U^I(1)$ which appears under the $SU(3) \supset SU(2) \times U^{II}(1) \supset U^I(1) \times U^{II}(1)$ decomposition or $\mathbb{Z}_2 \hookrightarrow U^{II}(1)$. Since the former is trivial, namely cannot break the $SU(3)$ appearing in eq. (3.5), only the latter is interesting for further investigation. This is realised as

$$
U_g^{(1)} = diag(-1,-1,1)\,. \tag{3.6}
$$

Indeed $(U_g^{(1)})^2 = \mathbb{1}_3$ as required by the $F^{S/R} \mapsto H$ homomorphism and $det(U_g^{(1)}) = 1$ since U_g is an H group element.

Then, the various components of the decomposition of $SU(3)$ irreps under $SU(2) \times U(1)$ acquire the underbraced phase factors in the following list

$$
\begin{aligned}
& SU(3) \supset SU(2) \times U(1) \\
& \mathbf{3}{=}\underbrace{\mathbf{1}_{(-2)}}_{(+1)} + \underbrace{\mathbf{2}_{(1)}}_{(-1)}\,, \\
& \mathbf{6}{=}\underbrace{\mathbf{1}_{(-4)}}_{(+1)} + \underbrace{\mathbf{2}_{(-1)}}_{(-1)} + \underbrace{\mathbf{3}_{(2)}}_{(+1)}\,, \\
& \mathbf{8}{=}\underbrace{\mathbf{1}_{(0)}}_{(+1)} + \underbrace{\mathbf{3}_{(0)}}_{(+1)} + \underbrace{\mathbf{2}_{(-3)}}_{(-1)} + \underbrace{\mathbf{2}_{(3)}}_{(-1)}\,.
\end{aligned}
\tag{3.7}
$$

Consequently the various components of the decomposition of E_6 irreps (3.5) under $F_4 \supset SU(3)\times SU^c(3) \supset (SU(2)\times U(1))\times SU^c(3)$ acquire the underbraced phase factors in the following list

$$
\begin{aligned}
&E_6 \supset SU(2)\times SU^c(3)\times U(1)\\
&\mathbf{27}=\underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)}\\
&\quad+\underbrace{(\mathbf{1},\mathbf{3})_{(-2)}}_{(+1)}+\underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)}+\underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(2)}}_{(+1)}+\underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(-1)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{1})_{(-3)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)},\\
&\mathbf{78}=\underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{8})_{(0)}}_{(+1)}\\
&\quad+\underbrace{(\mathbf{1},\mathbf{3})_{(-2)}}_{(+1)}+\underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(2)}}_{(+1)}+\underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(-4)}}_{(+1)}+\underbrace{(\mathbf{1},\mathbf{3})_{(4)}}_{(+1)}\\
&\quad+\underbrace{(\mathbf{2},\mathbf{1})_{(-3)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{1})_{(-3)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)}\\
&\quad+\underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)}+\underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(-1)}}_{(-1)}+\underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)}+\underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(-1)}}_{(-1)}\\
&\quad+\underbrace{(\overline{\mathbf{3}},\mathbf{3})_{(-2)}}_{(+1)}+\underbrace{(\mathbf{3},\overline{\mathbf{3}})_{(+2)}}_{(+1)}\,.
\end{aligned}
\tag{3.8}
$$

According to the discussion in section 3.1 the four-dimensional gauge group after the topological breaking is given by $K' = C_H(T^H)$. Counting the number of singlets under the action of $U_g^{(1)}$ in the **78** irrep. above suggests that $K' = SO(10)\times U(1)$, a fact which subsequently is determined according to the following decomposition of the **78** irrep.

$$
\begin{aligned}
&E_6 \supset SO(10)\times U(1)\\
&\mathbf{27}=\underbrace{\mathbf{1}_{(-4)}}_{(+1)}+\underbrace{\mathbf{10}_{(-2)}}_{(+1)}+\underbrace{\mathbf{16}_{(1)}}_{(-1)},\\
&\mathbf{78}=\underbrace{\mathbf{1}_{(0)}}_{(+1)}+\underbrace{\mathbf{45}_{(0)}}_{(+1)}+\underbrace{\mathbf{16}_{(-3)}}_{(-1)}+\underbrace{\overline{\mathbf{16}}_{(3)}}_{(-1)}\,.
\end{aligned}
\tag{3.9}
$$

It is interesting to note that although one would naively expect the E_6 gauge group to break further towards the SM one this is not the case. The singlets under the action of $U_g^{(1)}$ which occur in the adjoint irrep. of E_6 in eq. (3.8) add up to provide a larger final unbroken gauge symmetry, namely $SO(10)\times U(1)$.

Embedding (2): $\mathbb{Z}_2 \hookrightarrow SU(5)$ **of**

$$E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1)\,.$$

Similarly, I consider the maximal subgroups of E_6 and the corresponding decomposition of the fundamental and adjoint irreps

$$\begin{aligned} & E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1) \\ & \mathbf{27}{=}\mathbf{1}_{(0,-4)} + \mathbf{5}_{(2,-2)} + \overline{\mathbf{5}}_{(-2,-2)} + \mathbf{1}_{(-5,1)} + \overline{\mathbf{5}}_{(3,1)} + \mathbf{10}_{(-1,1)}\,, \\ & \mathbf{78}{=}\mathbf{1}_{(0,0)} + \mathbf{1}_{(0,0)} + \mathbf{24}_{(0,0)} + \mathbf{1}_{(-5,-3)} + \mathbf{1}_{(5,3)} \\ & \quad +\mathbf{5}_{(-3,3)} + \overline{\mathbf{5}}_{(3,-3)} + \mathbf{10}_{(4,0)} + \overline{\mathbf{10}}_{(-4,0)} + \mathbf{10}_{(-1,-3)} + \overline{\mathbf{10}}_{(1,3)}\,. \end{aligned} \tag{3.10}$$

My choice is to embed the $\mathbb{Z}_2$ discrete symmetry in an abelian $SU(5)$ subgroup in a way that is realised by the diagonal matrix

$$U_g^{(2)} = diag(-1,-1,1,1,1)\,. \tag{3.11}$$

Then the various components of the $SU(5)$ irreps decomposed under the $SU(2)\times SU(3)\times U(1)$ decomposition acquire the underbraced phase factors in the following list

$$\begin{aligned} & SU(5) \supset SU(2) \times SU(3) \times U(1) \\ & \mathbf{5}{=}\underbrace{(\mathbf{2},\mathbf{1})_{(3)}}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{3})_{(-2)}}_{(+1)}\,, \\ & \mathbf{10}{=}\underbrace{(\mathbf{1},\mathbf{1})_{(6)}}_{(+1)} + \underbrace{(\mathbf{1},\overline{\mathbf{3}})_{(-4)}}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{3})_{(1)}}_{(-1)}\,, \\ & \mathbf{24}{=}\underbrace{(\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{3},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{8})_{(0)}}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{3})_{(-5)}}_{(-1)} + \underbrace{(\mathbf{2},\overline{\mathbf{3}})_{(5)}}_{(-1)}\,. \end{aligned} \tag{3.12}$$

It can be proven, along the lines of the previous case (**1**), that $U_g^{(2)}$ leads to the breaking $E_6 \to SU(2) \times SU(6)$

$$\begin{aligned} & E_6 \supset SU(2) \times SU(6) \\ & \mathbf{27}{=}\underbrace{(\mathbf{2},\overline{\mathbf{6}})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{15})}_{(+1)}\,, \\ & \mathbf{78}{=}\underbrace{(\mathbf{3},\mathbf{1})}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{35})}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{20})}_{(-1)}\,, \end{aligned} \tag{3.13}$$

i.e. I find again an enhancement of the final gauge group as compared to the naively expected one.

Note that other choices of $\mathbb{Z}_2$ into $SU(5)$ embeddings either lead to trivial or to phenomenologically uninteresting results.

Table 3.1: Embeddings of $\mathbb{Z}_2$ discrete symmetry in E_6 GUT and its symmetry breaking patterns. $U_g^{(1)} = diag(-1,-1,1)$ *and* $U_g^{(2)} = diag(-1,-1,1,1,1)$ *as in text.*

Embedd.	U_g	$\mathbf{K}'$
1	$U_g^{(1)}$	$SO(10) \times U(1)$
2	$U_g^{(2)}$	$SU(2) \times SU(6)$
3	$\mathbb{1}_3 \otimes U_g^{(1)} \otimes \mathbb{1}_3$	$SU(2) \times SU(6)$

Embedding (3): $\mathbb{Z}_2 \hookrightarrow SU(3)$ **of** $E_6 \supset SU^w(3) \times SU(3) \times SU^c(3)$ **.**

I consider the maximal subgroup of E_6 and the corresponding decomposition of fundamental and adjoint irreps

$$\begin{aligned}
&E_6 \supset SU^w(3) \times SU(3) \times SU^c(3) \\
&\mathbf{27}{=}(\overline{\mathbf{3}},\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{3}) + (\mathbf{1},\overline{\mathbf{3}},\overline{\mathbf{3}})\,, \\
&\mathbf{78}{=}(\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) + (\mathbf{3},\mathbf{3},\overline{\mathbf{3}}) + (\overline{\mathbf{3}},\overline{\mathbf{3}},\mathbf{3})\,.
\end{aligned} \tag{3.14}$$

Furthermore I assume an $\mathbb{Z}_2 \hookrightarrow SU(3)$ embedding, which is realised by

$$U_g^{(3)} = (\mathbb{1}_3) \otimes diag(-1,-1,1) \otimes (\mathbb{1}_3)\,. \tag{3.15}$$

Although this choice of embedding is not enough to lead to the SM gauge group structure, the results will be usefull for the discussion of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ case which is presented in subsection 3.3.2. With the choice of embedding realised by the eq. (3.15) the second $SU(3)$ decomposes under $SU(2) \times U(1)$ as in eq. (3.7) and leads to the breaking (3.13), as before. As was mentioned in case (**1**) the choice of embedding $\mathbb{Z}_2 \hookrightarrow U^I(1)$, which appears under the decomposition $SU(3) \supset SU(2) \times U^{II}(1) \supset U^I(1) \times U^{II}(1)$ of eq. (3.14), cannot break the $SU(3)$ group factor and it is not an interesting case for further investigation.

In table 3.1 I summarise the above results, concerning the topologically induced symmetry breaking patterns of the E_6 gauge group.

3.3.2 The $\mathbb{Z}_2 \times \mathbb{Z}_2'$ case

Embedding (2′): $\mathbb{Z}_2 \hookrightarrow SO(10)$ **and** $\mathbb{Z}_2' \hookrightarrow SU(5)$ **of**

$$E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1)\,.$$

Here I embed the $\mathbb{Z}_2$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ discrete symmetry in the $SU(5)$ appearing under the decomposition $E_6 \supset SO(10) \times U(1) \supset SU(5) \times U(1) \times U(1)$ as in case (**2**) above.

Furthermore I embed the $\mathbb{Z}_2'$ discrete symmetry in the $SO(10)$ as

$$U_g' = -\mathbb{1}_{10}\,. \tag{3.16}$$

This leads to the breaking $E_6 \supset SU(2) \times SU(6)$ as before but with the signs of the phase factors, which appear in eq. (3.13), being reversed under the action of $U_g^{(2)} U_g'$.

Embedding (3′): $\mathbb{Z}_2 \hookrightarrow SU(3)$ **and** $\mathbb{Z}_2' \hookrightarrow SU^w(3)$ **of**

$$E_6 \supset SU^w(3) \times SU(3) \times SU^c(3)\,.$$

Here I embed the $\mathbb{Z}_2$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ discrete symmetry in the $SU(3)$ group factor appearing under the $E_6 \supset SU(3)^w \times SU(3) \times SU(3)^c$ as in case (**3**) above. Furthermore I embed the $\mathbb{Z}_2'$ discrete symmetry in the $SU(3)^w$ group factor in a similar way. Then the embedding (**3′**), which I discuss here, is realised by considering an element of the E_6 gauge group

$$U_g' U_g^{(3)} = diag(-1,-1,1) \otimes diag(-1,-1,1) \otimes (\mathbb{1}_3)\,, \tag{3.17}$$

which leads to the breaking $E_6 \to SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1)$ as it is clear from the following decomposition of **78** irrep.

$$\begin{aligned}
&E_6 \supset SO(10) \times U(1) \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1) \\
&E_6 \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1) \\
&\mathbf{78} = \underbrace{(\mathbf{1},\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{3},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{3},\mathbf{1},\mathbf{1})_{(0)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{15})_{(0)}}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{2},\mathbf{6})_{(0)}}_{(-1)} \\
&\quad + \underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})_{(-3)}}_{(-1)} + \underbrace{(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})_{(3)}}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{2},\mathbf{4})_{(3)}}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{(-3)}}_{(-1)}\,.
\end{aligned} \tag{3.18}$$

Furthermore the irrep. **27** of E_6 decomposes under the same breaking as

$$\begin{aligned}
&E_6 \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1) \\
&\mathbf{27} = \underbrace{(\mathbf{1},\mathbf{1},\mathbf{1})_{(4)}}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{2},\mathbf{1})_{(-2)}}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{6})_{(-2)}}_{(+1)} \\
&\quad + \underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})_{(1)}}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{(1)}}_{(-1)}\,.
\end{aligned} \tag{3.19}$$

In table 3.2 I summarise the above results, concerning the topologically induced symmetry breaking patterns of the E_6 gauge group.

Table 3.2: Embeddings of $\mathbb{Z}_2 \times \mathbb{Z}_2'$ discrete symmetries in E_6 GUT and its symmetry breaking patterns. *$U_g^{(1)} = diag(-1,-1,1)$ and $U_g^{(2)} = diag(-1,-1,1,1,1)$ as in text.*

Embedd.	U_g	U_g'	$\mathbf{K}'$
$\mathbf{2}'$	$U_g^{(2)}$	$-\mathbb{1}_{10}$	$SU(2) \times SU(6)$
$\mathbf{3}'$	$U_g^{(1)} \otimes \mathbb{1}_3 \otimes \mathbb{1}_3$	$\mathbb{1}_3 \otimes U_g^{(1)} \otimes \mathbb{1}_3$	$SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1)$

3.4 Symmetry breaking pattern of $SO(10)$-like GUTs

Here I determine the image, T^H, that each of the discrete symmetries of eq. (3.4) induces in the gauge group $H = SO(10)$. I consider embeddings of the $F^{S/R}$ discrete symmetries into abelian subgroups of $SO(10)$ GUTs and examine their topologically induced symmetry breaking patterns. The interesting $F^{S/R} \hookrightarrow SO(10)$ embeddings are those which potentially lead to SM gauge group structure, i.e.

$$SO(10) \supset SU(5) \times U^{II}(1) \supset SU^w(2) \times SU^c(3) \times U^I(1) \times U^{II}(1)\,.$$

3.4.1 The $\mathbb{Z}_2$ case

Embedding (1): $\mathbb{Z}_2 \hookrightarrow SU(5)$ of $SO(10) \supset SU(5) \times U(1)$.

In the present case I assume the maximal subgroup of $SO(10)$

$$\begin{aligned} &SO(10) \supset SU(5) \times U^{II}(1) \\ &\mathbf{10} = \mathbf{5}_{(2)} + \overline{\mathbf{5}}_{(-2)}\,, \\ &\mathbf{16} = \mathbf{1}_{(-5)} + \overline{\mathbf{5}}_{(3)} + \mathbf{10}_{(-1)}\,, \\ &\mathbf{45} = \mathbf{1}_{(0)} + \mathbf{24}_{(0)} + \mathbf{10}_{(4)} + \overline{\mathbf{10}}_{(-4)}\,, \end{aligned} \tag{3.20}$$

and embed a $\mathbb{Z}_2 \hookrightarrow SU(5)$ which is realised as in eq. (3.11). Then, the **5**, **10** and **24** irreps of $SU(5)$ under the $SU(5) \supset SU(2) \times SU(3) \times U(1)$ decomposition read as in eq. (3.12) and lead to the breaking $SO(10) \to SU^a(2) \times SU^b(2) \times SU(4)$

$$\begin{aligned} &SO(10) \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \\ &\mathbf{10} = \underbrace{(\mathbf{2},\mathbf{2},\mathbf{1})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{6})}_{(+1)}\,, \\ &\mathbf{16} = \underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})}_{(+1)}\,, \\ &\mathbf{45} = \underbrace{(\mathbf{3},\mathbf{1},\mathbf{1})}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{3},\mathbf{1})}_{(+1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{15})}_{(+1)} + \underbrace{(\mathbf{2},\mathbf{2},\mathbf{6})}_{(-1)}\,. \end{aligned} \tag{3.21}$$

Table 3.3: Embedding of $\mathbb{Z}_2$ discrete symmetry in $SO(10)$ GUT and its symmetry breaking pattern. $U_g^{(2)} = diag(-1,-1,1,1,1)$ *as in text.*

Embedd.	U_g	$\mathbf{K}'$
1	$U_g^{(2)}$	$SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)$

Table 3.4: Embedding of $\mathbb{Z}_2 \times \mathbb{Z}_2'$ discrete symmetries in $SO(10)$ GUT and its symmetry breaking pattern. $U_g^{(2)} = diag(-1,-1,1,1,1)$ *as in text.*

Embedd.	U_g	U_g'	$\mathbf{K}'$
$\mathbf{1}'$	$U_g^{(2)}$	$-\mathbb{1}_{10}$	$SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)$

Note again that although one would naively expect the $SO(10)$ gauge group to break towards SM, this is not the case.

For completeness in table 3.3 I present the above case.

3.4.2 The $\mathbb{Z}_2 \times \mathbb{Z}_2'$ case.

Embedding ($\mathbf{1}'$): $\mathbb{Z}_2 \hookrightarrow SU(5)$ and $\mathbb{Z}_2' \hookrightarrow SO(10)$ of $SO(10) \supset SU(5) \times U(1)$.

Note that a second $\mathbb{Z}_2$ cannot break the $K' = SU^a(2) \times SU^b(2) \times SU(4)$ further. However by choosing the non-trivial embedding $U_g' = -\mathbb{1}_{10}$ of $\mathbb{Z}_2$ in the $SO(10)$ the phase factors appearing in eq. (3.21) have their signs reversed under the action of $U_g^{(2)} U_g'$.

Again in table 3.4 I present the above case.

3.5 Reduction of $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac over $SO(7)/SO(6)$ revisited

In section 2.4 an illuminating example of application of the CSDR scheme was presented. I discussed there the dimensional reduction of a $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory over the coset space $S/R = SO(7)/SO(6) \sim S^6$. The gauge group of the resulting four-dimensional theory was found to be $H = SO(10)$. The theory was provided with scalars belonging in the **10** of $SO(10)$ and with two copies of chiral fermions transforming as $\mathbf{16}_L$ under the same gauge group. The scalars of the theory being in the fundamental rep. of the H gauge group are not able to provide the appropriate superstrong symmetry

breaking towards the SM. However one can use the freely acting discrete symmetries, $F^{S/R}$, of the coset space $S/R = SO(7)/SO(6)$, assume non-trivial topological properties for it and break the gauge symmetry further according to section 3.1.

To be more specific the $F^{S/R}$ discrete symmetries of the coset space $SO(7)/SO(6)$ (case '**a**' in table 2.1) are that of Weyl, $\mathrm{W} = \mathbb{Z}_2$, and the center of S, $\mathrm{Z(S)} = \mathbb{Z}_2$. As it was explained in section 3.2 the use of $\mathrm{Z}(S)$ is excluded. On the other hand, according to the discussion in subsection 3.4.1 the W discrete symmetry leads to a four-dimensional theory with the following gauge group

$$K' = C_H(T^H) = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)\,.$$

According to section 3.1, the surviving field content has to be invariant under the combined action of the considered discrete symmetry itself, $F^{S/R}$, and its induced image in the H gauge group, T^H [eq. (3.3)]. Using the $\mathrm{W} = \mathbb{Z}_2$ discrete symmetry, the irrep. **10** of $SO(10)$ decomposes under the $SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)$ is given in eq. (3.21),

$$\begin{aligned} &SO(10) \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \\ &\mathbf{10}{=}\underbrace{(\mathbf{2},\mathbf{2},\mathbf{1})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{1},\mathbf{6})}_{(+1)}\,. \end{aligned} \tag{3.22}$$

Then, recalling that the $R = SO(6)$ content of the vector $B_0 = SO(7)/SO(6)$ is invariant under the action of W (table 2.1), I conclude that the four-dimensional theory contains scalars transforming according to

$$(\mathbf{1},\mathbf{1},\mathbf{6})$$

of K'. Similarly, the irrep. **16** of $SO(10)$ decomposes under the K' as

$$\begin{aligned} &SO(10) \supset SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \\ &\mathbf{16}{=}\underbrace{(\mathbf{2},\mathbf{1},\mathbf{4})}_{(-1)} + \underbrace{(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})}_{(+1)}\,. \end{aligned} \tag{3.23}$$

In this case the spinor of the tangent space of $SO(7)/SO(6)$ decomposed under $R = SO(6)$ is obviously **4**. Then, since the W transformation properties for the spinor is $\mathbf{4}{\leftrightarrow}\overline{\mathbf{4}}$ (table 2.1), the fermion content of the four-dimensional theory transforms as

$$(\mathbf{2},\mathbf{1},\mathbf{4})_L - (\mathbf{2},\mathbf{1},\mathbf{4})'_L \qquad \text{and} \qquad (\mathbf{1},\mathbf{2},\mathbf{4})_L + (\mathbf{1},\mathbf{2},\mathbf{4})'_L \tag{3.24}$$

under K'.

According to the discussion in subsection 2.2.3, dimensional reduction over the $\frac{SO(7)}{SO(6)}$ symmetric coset space leads to a four-dimensional potential with spontaneously symmetry breaking form. However, since the four-dimensional scalar fields belong in the $(\mathbf{1},\mathbf{1},\mathbf{6})$ under the K' gauge group obtaining a v.e.v. break the $SU(3)$ colour. Therefore, employing the W discrete symmetry is not an interesting case to investigate further.

On the other hand, if I use the $\mathrm{W} \times \mathrm{Z}(S) = \mathbb{Z}_2 \times \mathbb{Z}_2$ discrete symmetry, the Wilson flux breaking mechanism leads again to the Pati-Salam gauge group, K' (see subsection 3.4.2). However in this case, all the underbraced phase factors of eqs. (3.22)

and (3.23) are multiplied by -1. As before, the surviving fields in four dimensions have to be invariant under the action $F^{S/R} \oplus T^H$. Therefore the four-dimensional theory now contains scalars transforming according to

$$(\mathbf{2}, \mathbf{2}, \mathbf{1})$$

of K', and two copies of chiral fermions transforming as in eq. (3.24) but with the signs of the linear combinations reversed.

Note that, if only the CSDR mechanism was applied, the final gauge group would be (see section 2.4)

$$\mathcal{H} = C_{E_8}(SO(7)) = SO(9)\,,$$

i.e. the $\mathbf{10}$ of $SO(10)$ would obtain a v.e.v. and lead to the spontaneous symmetry breaking

$$\begin{aligned} &SO(10) \to SO(9) \\ &\mathbf{10} = \langle \mathbf{1} \rangle + \mathbf{9}\,. \end{aligned} \tag{3.25}$$

However now I employ the Wilson flux breaking mechanism which breaks the gauge symmetry further in higher dimensions. It is instructive to understand the spontaneous symmetry breaking indicated in eq. (3.25) in this context too. A straightforward examination of the gauge group structure and the representations of the scalars that are involved, suggests the breaking indicated in eq. (3.22) is realised in the present context as

$$\begin{aligned} SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \to & SU^{diag}(2) \times SU(4) \\ (\mathbf{2}, \mathbf{2}, \mathbf{1}) \; = & \langle (\mathbf{1}, \mathbf{1}) \rangle + (\mathbf{3}, \mathbf{1})\,, \end{aligned} \tag{3.26}$$

i.e. the final gauge group of the four-dimensional theory is

$$K = SU^{diag}(2) \times SU(4)\,.$$

Accordingly, the fermions transform as

$$(\mathbf{2}, \mathbf{4})_L + (\mathbf{2}, \mathbf{4})'_L \qquad \text{and} \qquad (\mathbf{2}, \mathbf{4})_L - (\mathbf{2}, \mathbf{4})'_L$$

under K.

Chapter 4

Classification of semi-realistic particle physics models

Here starting from an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory defined in ten dimensions, I provide a complete classification of semi-realistic particle physics models resulting from CSDR and a subsequent application of the Wilson flux breaking mechanism. According to our requirements in section 2.3 the dimensional reduction of this theory over the coset spaces S/R which are listed in the first column of tables 2.1 and 2.2, leads to anomaly free E_6 and $SO(10)$ GUTs in four dimensions. Recall also that the four-dimensional surviving scalars transform in the fundamental of the resulting gauge group and are not suitable for the superstrong symmetry breaking towards the SM. One way out was discussed in chapter 3, namely the Wilson flux breaking mechanism. Here, I investigate to which extent applying both methods, CSDR and Wilson flux breaking mechanism one can obtain reasonable low energy models.

4.1 Dimensional reduction over symmetric coset spaces

Here, I consider all the possible embeddings $E_8 \supset SO(6) \supset R$ for the six-dimensional *symmetric* coset spaces, S/R, listed in the first column of table 2.1*. These embeddings are presented in figure 4.1. It is worth noting that in all cases the dimensional reduction of the initial gauge theory leads to an $SO(10)$ GUT according to the concluding remarks in subsection 2.2.2. The result of my examination in the present section is that the additional use of Wilson flux breaking mechanism summarize leads to four-dimensional theories of Pati-Salam type. In the following subsections 4.1.1 - 4.1.5 I present details of

*I have excluded the study of dimensional reduction over the $Sp(4)/(SU(2) \times U(1))_{max}$ coset space which does not admit fermions.

$$
\begin{tikzcd}
& SU(4)\ (\mathbf{1}) & & \\
SO(6)\sim SU(4) \arrow[ur] \arrow[r] \arrow[dr] & SU(3)\times U(1)\ (\mathbf{2}) \arrow[r] & SU^a(2)\times U(1)\times U(1)\ (\mathbf{3}) \arrow[r] & U(1)\times U(1)\times U(1)\ (\mathbf{4}) \\
& SU^a(2)\times SU^b(2)\times U(1)\ (\mathbf{5}) \arrow[r] & SU^a(2)\times U(1)\times U(1)\ (\mathbf{6}) \arrow[r] & U(1)\times U(1)\times U(1)\ (\mathbf{7})
\end{tikzcd}
$$

Figure 4.1: *Possible $E_8 \supset SO(6) \supset R$ embeddings for the symmetric coset spaces, S/R, of table 2.1.*

my study and the corresponding results which I in tables A.1 and A.2[†].

4.1.1 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = SO(7)/SO(6)$. (Case 1a)

I consider the dimensional reduction of an $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory over the six-dimensional sphere, $S^6 \sim SO(7)/SO(6)$. This model was presented in section 2.4 as an example of applying the CSDR scheme. The assumed embedding of R into $G = E_8$ is the one suggested by the maximal subgroup $E_8 \supset SO(6) \times SO(10)$ and it is denoted as $(\mathbf{1})$ in figure 4.1. In section 3.5 I applied the Hosotani breaking mechanism to obtain a reasonable low energy model. I summarize the results of my examination in the first row of tables A.1 and A.2.

4.1.2 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = SU(4)/(SU(3) \times U(1))$. (Case 2b)

I consider again Weyl fermions belonging in the adjoint of $G = E_8$ and the embedding of $R = SU(3) \times U(1)$ into E_8 suggested by the decomposition[‡].

$$
E_8 \supset SO(16) \supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset SU(3) \times U^I(1) \times SO(10)
$$

$$
\begin{aligned}
E_8 \ &\supset (SU(3) \times U^I(1)) \times SO(10) \\
\mathbf{248} &= (\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{45})_{(0)} + (\mathbf{8},\mathbf{1})_{(0)} + (\mathbf{3},\mathbf{10})_{(-2)} + (\overline{\mathbf{3}},\mathbf{10})_{(2)} \\
&\quad + (\mathbf{3},\mathbf{1})_{(4)} + (\overline{\mathbf{3}},\mathbf{1})_{(-4)} + (\mathbf{1},\mathbf{16})_{(-3)} + (\mathbf{1},\overline{\mathbf{16}})_{(3)} \\
&\quad + (\mathbf{3},\mathbf{16})_{(1)} + (\overline{\mathbf{3}},\overline{\mathbf{16}})_{(-1)} \, .
\end{aligned}
\tag{4.1}
$$

[†]For convenience I label the cases examined in the following subsections as 'Case No.x'. 'No' denotes the embedding $R \hookrightarrow E_8$ and the 'x' the coset space I use. The same label is also used in tables A.1 and A.2.

[‡]This decomposition is in accordance with the Slansky tables [57] but with opposite $U(1)$ charge.

If only the the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU(3) \times U^I(1)) = SO(10) \Big(\times U^I(1) \Big) \,,$$

where the additional $U(1)$ factor in the parenthesis corresponds to a global symmetry, according to the concluding remarks in subsection 2.2.2. The $R = SU(3) \times U^I(1)$ content of the vector and spinor of $B_0 = S/R = SU(4)/(SU(3) \times U(1))$ can be read in the last two columns of table 2.1. Then according to the CSDR rules, the theory would contain scalars belonging in the $\mathbf{10}_{(-2)}$, $\mathbf{10}_{(2)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(3)}$ and $\mathbf{16}_{L(-1)}$ under the same gauge group.

The freely acting discrete symmetries of the coset space $SU(4)/(SU(3) \times U(1))$ are not included in the list (3.4) of those ones that are worth to be examined further.

4.1.3 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = SU(3)/(SU(2) \times U(1)) \times (SU(2)/U(1))$. (Cases 3d, 6d)

I consider again Weyl fermions belonging in the adjoint of $G = E_8$ and the following decomposition

$$\begin{aligned} E_8 \supset SO(16) \supset\, & SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset (SU'(3) \times U^{II}(1)) \times SO(10) \\ & \supset (SU^a(2) \times U^I(1) \times U^{II}(1)) \times SO(10) \end{aligned} \tag{4.2}$$

or

$$\begin{aligned} E_8 \supset SO(16) \supset\, & SO(6) \times SO(10) \backsim SU(4) \times SO(10) \\ & \supset (SU^a(2) \times SU^b(2) \times U^{II}(1)) \times SO(10) \\ & \supset (SU^a(2) \times U^I(1) \times U^{II}(1)) \times SO(10) \,. \end{aligned} \tag{4.3}$$

In both cases I can properly redefine the $U(1)$ charges, and consequently choose an embedding of $R = SU(2) \times U^I(1) \times U^{II}(1)$ into E_8 as follows

$$\begin{aligned} E_8 \;& \supset (SU^a(2) \times U^{I'}(1) \times U^{II'}(1)) \times SO(10) \\ \mathbf{248} &= (\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{3},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{45})_{(0,0)} \\ & + (\mathbf{1},\mathbf{1})_{(-2b,0)} + (\mathbf{1},\mathbf{1})_{(2b,0)} + (\mathbf{2},\mathbf{1})_{(-b,2a)} + (\mathbf{2},\mathbf{1})_{(b,-2a)} \\ & + (\mathbf{2},\mathbf{1})_{(-b,-2a)} + (\mathbf{2},\mathbf{1})_{(b,2a)} + (\mathbf{1},\mathbf{10})_{(0,-2a)} + (\mathbf{1},\mathbf{10})_{(0,2a)} \\ & + (\mathbf{2},\mathbf{10})_{(b,0)} + (\mathbf{2},\mathbf{10})_{(-b,0)} + (\mathbf{1},\mathbf{16})_{(b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(-b,a)} \\ & + (\mathbf{1},\mathbf{16})_{(-b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(b,a)} + (\mathbf{2},\mathbf{16})_{(0,a)} + (\mathbf{2},\overline{\mathbf{16}})_{(0,-a)} \,. \end{aligned} \tag{4.4}$$

Here, a and b are the $U(1)$ charges of vector and fermion content of the coset space $B_0 = S/R = SU(3)/(SU(2) \times U^{I'}(1)) \times (SU(2)/U^{II'}(1))$, shown in the last two columns

of table 2.1 (case '**d**'). Then, if only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU^a(2) \times U^{I'}(1) \times U^{II'}(1)) = SO(10)\left(\times U^{I'}(1) \times U^{II'}(1)\right),$$

where the additional $U(1)$ factors in the parenthesis correspond to global symmetries. According to the CSDR rules, the four-dimensional model would contain scalars belonging in $\mathbf{10}_{(0,-2a)}$, $\mathbf{10}_{(0,2a)}$, $\mathbf{10}_{(b,0)}$ and $\mathbf{10}_{(-b,0)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(b,-a)}$, $\mathbf{16}_{L(-b,-a)}$ and $\mathbf{16}_{L(0,a)}$ under the same gauge group.

The freely acting discrete symmetries of the coset space under discussion, are the center of S, Z(S) $= \mathbb{Z}_3 \times \mathbb{Z}_2$ and the Weyl symmetry, W $= \mathbb{Z}_2$. Then according to the list (3.4) the interesting cases to be examined further are the following two.

In the first case I employ the W $= \mathbb{Z}_2$ discrete symmetry which leads to a four-dimensional theory with gauge symmetry group

$$K' = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4)\left(\times U^{I'}(1) \times U^{II'}(1)\right).$$

Similarly to the case discussed in subsection 4.1.1, the surviving scalars transform as

$$(\mathbf{1},\mathbf{1},\mathbf{6})_{(b,0)} \qquad \text{and} \qquad (\mathbf{1},\mathbf{1},\mathbf{6})_{(-b,0)} \tag{4.5}$$

under K' which are the only ones that are invariant under the action of W (table 2.1). Furthermore, taking into account the W transformation properties listed in the last column of table 2.1, as well as the decomposition of $\mathbf{16}$ irrep. of $SO(10)$ under $SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)$ [see eq. (3.23)], I conclude that the four-dimensional fermions transform as

$$\begin{array}{cc}
\multicolumn{2}{c}{(\mathbf{2},\mathbf{1},\mathbf{4})_{L(b,-a)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(b,-a)}\,,} \\
\multicolumn{2}{c}{(\mathbf{1},\mathbf{2},\mathbf{4})_{L(b,-a)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(b,-a)}\,,} \\
(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-b,-a)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-b,-a)}\,, & (\mathbf{2},\mathbf{1},\mathbf{4})_{L(0,a)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(0,a)}\,, \\
(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-b,-a)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-b,-a)}\,, & (\mathbf{1},\mathbf{2},\mathbf{4})_{L(0,a)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(0,a)}\,,
\end{array} \tag{4.6}$$

under K'.

Once more I have spontaneous symmetry breaking (since the coset space is symmetric) which breaks the $SU(3)$-colour [since the scalars transform as in (4.5) under the K' gauge group]. Therefore, employing the W discrete symmetry is not an interesting case for further investigation.

In the second case I use the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the W $\times$ Z(S) combination of discrete symmetries. The surviving scalars of the four-dimensional theory belong in the

$$(\mathbf{2},\mathbf{2},\mathbf{1})_{(b,0)}\,, \qquad \text{and} \qquad (\mathbf{2},\mathbf{2},\mathbf{1})_{(-b,0)}$$

of the K' gauge group which remain the same as before. The fermions, on the other hand, transform as those in eq. (4.6) but with the signs of the linear combinations reversed.

The final gauge group after the spontaneous symmetry breaking of the theory is found to be

$$K = SU^{diag}(2) \times SU(4) \Big(\times U^{I'}(1) \times U^{II'}(1) \Big) .$$

and its fermions transform as

$$\begin{array}{cc} \multicolumn{2}{c}{(\mathbf{2},\mathbf{4})_{(b,-a)} + (\mathbf{2},\mathbf{4})'_{(b,-a)} \,,} \\ \multicolumn{2}{c}{(\mathbf{2},\mathbf{4})_{(b,-a)} - (\mathbf{2},\mathbf{4})'_{(b,-a)} \,,} \\ (\mathbf{2},\mathbf{4})_{(-b,-a)} + (\mathbf{2},\mathbf{4})'_{(-b,-a)} \,, & (\mathbf{2},\mathbf{4})_{(0,a)} + (\mathbf{2},\mathbf{4})'_{(0,a)} \,, \\ (\mathbf{2},\mathbf{4})_{(-b,-a)} - (\mathbf{2},\mathbf{4})'_{(-b,-a)} \,, & (\mathbf{2},\mathbf{4})_{(0,a)} - (\mathbf{2},\mathbf{4})'_{(0,a)} \,, \end{array} \tag{4.7}$$

under K.

4.1.4 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = (SU(2)/U(1))^3$. (Cases 4f, 7f)

I consider again Weyl fermions belonging in the adjoint of $G = E_8$ and the following decomposition

$$\begin{aligned} E_8 \supset SO(16) &\supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \supset SU'(3) \times U^{III}(1) \times SO(10) \\ &\supset (SU^a(2) \times U^{II}(1) \times U^{III}(1)) \times SO(10) \\ &\supset SO(10) \times U^{I}(1) \times U^{II}(1) \times U^{III}(1) \end{aligned} \tag{4.8}$$

or

$$\begin{aligned} E_8 \supset SO(16) &\supset SO(6) \times SO(10) \backsim SU(4) \times SO(10) \\ &\supset (SU^a(2) \times SU^b(2) \times U^{III}(1)) \times SO(10) \\ &\supset SO(10) \times U^{I}(1) \times U^{II}(1) \times U^{III}(1) \,. \end{aligned} \tag{4.9}$$

In both cases I can properly redefine the $U(1)$ charges, and consequently choose an embedding of $R = SU(2) \times U^I(1) \times U^{II}(1)$ into E_8 as follows

$$\begin{aligned} E_8 \; &\supset SO(10) \times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1) \\ \mathbf{248} &= \mathbf{1}_{(0,0,0)} + \mathbf{1}_{(0,0,0)} + \mathbf{1}_{(0,0,0)} + \mathbf{45}_{(0,0,0)} \\ &\quad + \mathbf{1}_{(-2a,2b,0)} + \mathbf{1}_{(2a,-2b,0)} + \mathbf{1}_{(-2a,-2b,0)} + \mathbf{1}_{(2a,2b,0)} \\ &\quad + \mathbf{1}_{(-2a,0,-2c)} + \mathbf{1}_{(2a,0,2c)} + \mathbf{1}_{(0,-2b,-2c)} + \mathbf{1}_{(0,2b,2c)} \\ &\quad + \mathbf{1}_{(-2a,0,2c)} + \mathbf{1}_{(2a,0,-2c)} + \mathbf{1}_{(0,-2b,2c)} + \mathbf{1}_{(0,2b,-2c)} \\ &\quad + \mathbf{10}_{(0,0,2c)} + \mathbf{10}_{(0,0,-2c)} + \mathbf{10}_{(0,2b,0)} + \mathbf{10}_{(0,-2b,0)} \\ &\quad + \mathbf{10}_{(2a,0,0)} + \mathbf{10}_{(-2a,0,0)} + \mathbf{16}_{(a,b,c)} + \overline{\mathbf{16}}_{(-a,-b,-c)} \\ &\quad + \mathbf{16}_{(-a,-b,c)} + \overline{\mathbf{16}}_{(a,b,-c)} + \mathbf{16}_{(-a,b,-c)} + \overline{\mathbf{16}}_{(a,-b,c)} \\ &\quad + \mathbf{16}_{(a,-b,-c)} + \overline{\mathbf{16}}_{(-a,b,c)} \,. \end{aligned} \tag{4.10}$$

Then, if only the CSDR mechanism was applied, the four-dimensional gauge group would be

$$H = C_{E_8}(U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1)) = SO(10) \Big(\times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1) \Big) .$$

The same comment as in the previous cases holds for the additional $U(1)$ factors in the parenthesis. The $R = U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1)$ content of vector and spinor of $B_0 = S/R = (SU(2)/U^{I'}(1)) \times (SU(2)/U^{II'}(1)) \times (SU(2)/U^{III'}(1))$ can be read in the last two columns of table 2.1 (case '**f**'). According to the CSDR rules then, the resulting four-dimensional theory would contain scalars belonging in $\mathbf{10}_{(2a,0,0)}$, $\mathbf{10}_{(-2a,0,0)}$, $\mathbf{10}_{(0,2b,0)}$, $\mathbf{10}_{(0,-2b,0)}$, $\mathbf{10}_{(0,0,2c)}$ and $\mathbf{10}_{(0,0,-2c)}$ of H and two copies of fermions transforming as $\mathbf{16}_{L(a,b,c)}$, $\mathbf{16}_{L(-a,-b,c)}$, $\mathbf{16}_{L(-a,b,-c)}$ and $\mathbf{16}_{L(a,-b,-c)}$ under the same gauge group.

The freely acting discrete symmetries, $F^{S/R}$, of the coset space $(SU(2)/U(1))^3 \sim (S^2)^3$ are the center of S, Z(S) $= (\mathbb{Z}_2)^3$ and the Weyl discrete symmetry, W $= (\mathbb{Z}_2)^3$. Then according to the list (3.4) the interesting cases to be examined further are the following.

First, let me mod out the $(S^2)^3$ coset space by the $\mathbb{Z}_2 \subset$ W and consider the multiply connected manifold $S^2/\mathbb{Z}_2 \times S^2 \times S^2$. Then, the resulting four-dimensional gauge group will be

$$K' = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \Big(\times U^{I'}(1) \times U^{II'}(1) \times U^{III'}(1) \Big) .$$

The four-dimensional theory will contain scalars which belong in

$$\begin{array}{ll} (\mathbf{1},\mathbf{1},\mathbf{6})_{(0,2b,0)} , & (\mathbf{1},\mathbf{1},\mathbf{6})_{(0,-2b,0)} , \\ (\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,2c)} , & (\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,-2c)} \end{array}$$

of K'; these are the only ones that are invariant under the action of the considered $\mathbb{Z}_2 \subset$ W. However, linear combinations between the two copies of the CSDR-surviving left-handed fermions have no definite properties under the abelian factors of the K' gauge group and they do not survive. As a result, the model is not an interesting case for further investigation.

Second, if I employ the $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset$ W discrete symmetry and consider the manifold $S^2/\mathbb{Z}_2 \times S^2/\mathbb{Z}_2 \times S^2$, the resulting four-dimensional theory has the same gauge symmetry as before, i.e. K'. Similarly as before, scalars transform as

$$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,2c)} , \qquad (\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,-2c)}$$

under K'. However, no fermions survive in the four-dimensional theory and the model is again not an interesting case to examine further.

Finally, if I employ the $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset$ W $\times$ Z(S) discrete symmetry, the four-dimensional theory contains scalars which belong in

$$\begin{array}{ll} (\mathbf{2},\mathbf{2},\mathbf{1})_{(0,2b,0)} , & (\mathbf{2},\mathbf{2},\mathbf{1})_{(0,-2b,0)} , \\ (\mathbf{2},\mathbf{2},\mathbf{1})_{(0,0,2c)} , & (\mathbf{2},\mathbf{2},\mathbf{1})_{(0,0,-2c)} \end{array}$$

of K' but no fermions. The model is again not an interesting case for further study.

Therefore although the above studied cases have been obtained using discrete symmetries which are included in the list (3.4), no fermion fields survive in the four-dimensional theory. The reason is that I employ here only a subgroup of the Weyl discrete symmetry $\mathrm{W} = (\mathbb{Z}_2)^3$ and I cannot form linear combinations among the two copies of the CSDR-surviving left-handed fermions which are invariant under eq. (3.3). The use of the whole W discrete symmetry, on the other hand, would lead to four-dimensional theories with smaller gauge symmetry than the one of SM.

4.1.5 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = Sp(4)/(SU(2) \times SU(2)) \times (SU(2)/U(1))$. (Case 5e)

Finally, I consider Weyl fermions in the adjoint of $G = E_8$ and the embedding of $R = SU(2) \times SU(2) \times U(1)$ into E_8 suggested by

$$\begin{aligned} E_8 \supset SO(16) \supset\ & SO(6) \times SO(10) \backsim SU(4) \times SO(10) \\ & \supset (SU^a(2) \times SU^b(2) \times U^I(1)) \times SO(10) \end{aligned}$$

$$\begin{aligned} E_8 \ &\supset (SU^a(2) \times SU^b(2) \times U^I(1)) \times SO(10) \\ \mathbf{248} &= (\mathbf{1},\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{1},\mathbf{45})_{(0)} + (\mathbf{3},\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{3},\mathbf{1})_{(0)} \\ &+ (\mathbf{2},\mathbf{2},\mathbf{1})_{(2)} + (\mathbf{2},\mathbf{2},\mathbf{1})_{(-2)} + (\mathbf{1},\mathbf{1},\mathbf{10})_{(2)} + (\mathbf{1},\mathbf{1},\mathbf{10})_{(-2)} \\ &+ (\mathbf{2},\mathbf{2},\mathbf{10})_{(0)} + (\mathbf{2},\mathbf{1},\mathbf{16})_{(1)} + (\mathbf{2},\mathbf{1},\overline{\mathbf{16}})_{(-1)} \\ &+ (\mathbf{1},\mathbf{2},\mathbf{16})_{(-1)} + (\mathbf{1},\mathbf{2},\overline{\mathbf{16}})_{(1)} . \end{aligned} \tag{4.11}$$

If only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU^a(2) \times SU^b(2) \times U^I(1)) = SO(10) \left(\times U^I(1) \right) .$$

The $R = SU^a(2) \times SU^b(2) \times U^{II}(1)$ content of vector and spinor of $B_0 = S/R = Sp(4)/(SU^a(2) \times SU^b(2)) \times (SU(2)/U(1))$ can be read in the last two columns of table 2.1. According to the CSDR rules the resulting four-dimensional theory would contain scalars belonging in $\mathbf{10}_{(0)}$, $\mathbf{10}_{(2)}$ and $\mathbf{10}_{(-2)}$ of H and two copies of chiral fermions transforming as $\mathbf{16}_{L(1)}$ and $\mathbf{16}_{L(-1)}$ under the same gauge group.

The freely acting discrete symmetries of the coset space $(Sp(4)/SU(2) \times SU(2)) \times (SU(2)/U(1))$ (case '**e**' in table 2.1), are the center of S, $\mathrm{Z(S)} = (\mathbb{Z}_2)^2$ and the Weyl discrete symmetry, $\mathrm{W} = (\mathbb{Z}_2)^2$. According to the list (3.4) the interesting cases to be examined further are the following.

First, if I employ the Weyl discrete symmetry, $\mathrm{W} = (\mathbb{Z}_2)^2$, leads to a four-dimensional theory with a gauge symmetry group

$$K' = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \left(\times U^I(1) \right) .$$

The surviving scalars of the theory belong in

$$(\mathbf{2},\mathbf{2},\mathbf{1})_{(0)}$$

of K', whereas its fermions transform as

$$\begin{array}{ll} (\mathbf{2},\mathbf{1},\mathbf{4})_{L(1)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(1)}\,, & (\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1)}\,, \\ (\mathbf{1},\mathbf{2},\mathbf{4})_{L(1)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(1)}\,, & (\mathbf{1},\mathbf{2},\mathbf{4})_{L(-1)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-1)} \end{array} \tag{4.12}$$

under K'.

Second, if I employ a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the $\mathrm{W} \times \mathrm{Z}(S)$ combination of discrete symmetries, leads to a four-dimensional model with scalars belonging in $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0)}$ and fermions transforming as in eq. (4.12) but with the signs of the linear combinations reversed.

Finally, in table A.2 I also report the less interesting case $\mathbb{Z}_2 \subseteq \mathrm{W}$.

Concerning the spontaneous symmetry breaking of theory, note that for the interesting cases of the $\mathrm{W} = (\mathbb{Z}_2)^2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{W} \times \mathrm{Z}(S)$ discrete symmetries, the final unbroken gauge group is found to be

$$K = SU^{diag}(2) \times SU(4) \Big(\times U(1) \Big)\,.$$

Then, for the case of W discrete symmetry, the fermions of the model transform as

$$\begin{array}{ll} (\mathbf{2},\mathbf{4})_{(1)} - (\mathbf{2},\mathbf{4})'_{(1)}\,, & (\mathbf{2},\mathbf{4})_{(-1)} - (\mathbf{2},\mathbf{4})'_{(-1)}\,, \\ (\mathbf{2},\mathbf{4})_{(1)} + (\mathbf{2},\mathbf{4})'_{(1)}\,, & (\mathbf{2},\mathbf{4})_{(-1)} + (\mathbf{2},\mathbf{4})'_{(-1)}\,, \end{array} \tag{4.13}$$

under K, whereas for the case of $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{W} \times \mathrm{Z}(S)$ the fermions belong in similar linear combinations as above but with their signs reversed.

4.2 Dimensional reduction over non-symmetric coset spaces

According to the discussion in section 2.3 I have to consider all the possible embeddings $E_8 \supset SO(6) \supset R$, for the six-dimensional *non-symmetric* cosets, S/R, of table 2.2. It is worth noting that the embedding of R in all cases of six-dimensional non-symmetric cosets are obtained by the following chain of maximal subgroups of $SO(6)$

$$SO(6) \sim SU(4) \supset SU(3) \times U(1) \supset SU^a(2) \times U(1) \times U(1) \supset U(1) \times U(1) \times U(1)\,. \tag{4.14}$$

It is also important to recall from the discussion in sections 2.2.2 and 2.3 that in all these cases the dimensional reduction of the initial gauge theory leads to an E_6 GUT. The result of my examination in the present section is that the additional use of the Wilson

flux breaking mechanism leads to four-dimensional gauge theories based on three different varieties of groups, namely $SO(10) \times U(1)$, $SU(2) \times SU(6)$ or $SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1)$. In the following subsections 4.2.1 - 4.2.3 I present details of my examination and summarize the corresponding results in tables A.3 and A.4[§].

4.2.1 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = G_2/SU(3)$. (Case 2a′)

I consider Weyl fermions belonging in the adjoint of $G = E_8$ and identify the R with the $SU(3)$ appearing in the decomposition (4.1). Then, if only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(SU(3)) = E_6 \,,$$

i.e. it appears an enhancement of the gauge group, a fact which was noticed earlier in several examples in subsections 3.3.1 and 3.3.2. This observation suggests that I could have considered the following more obvious embedding of $R = SU(3)$ into E_8,

$$\begin{aligned} E_8 \;&\supset SU'(3) \times E_6 \\ \mathbf{248} &= (\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{78}) + (\mathbf{3},\mathbf{27}) + (\overline{\mathbf{3}},\overline{\mathbf{27}}) \,. \end{aligned} \tag{4.15}$$

The $R = SU(3)$ content of vector and spinor of $B_0 = S/R = G_2/SU(3)$ coset is $\mathbf{3} + \overline{\mathbf{3}}$ and $\mathbf{1} + \mathbf{3}$, respectively. According to the CSDR rules, the four-dimensional theory would contain scalars belonging in $\mathbf{27}$ and $\overline{\mathbf{27}}$ of H, two copies of chiral fermions transforming as $\mathbf{27}_L$ under the same gauge group and a set of fermions in the $\mathbf{78}$ irrep., since the dimensional reduction over non-symmetric coset preserves the supersymmetric spectrum [28–30].

The freely acting discrete symmetry, $F^{S/R}$, of the coset space $G_2/SU(3)$ is the Weyl, $\mathrm{W} = \mathbb{Z}_2$ (case '**a**′' in table 2.2). Then, following the discussion in subsection 3.3.1, the Wilson flux breaking mechanism leads to a four-dimensional theory either with gauge group

$$(i) \qquad K'^{(\mathbf{1})} = C_H(T^H) = SO(10) \times U(1) \,, \tag{4.16}$$

in case I embed the $\mathbb{Z}_2$ into the E_6 gauge group as in the embedding (**1**) of subsection 3.3.1, or

$$(ii) \qquad K'^{(\mathbf{2},\mathbf{3})} = C_H(T^H) = SU(2) \times SU(6) \,, \tag{4.17}$$

in case I choose to embed the discrete symmetry as in the embeddings (**2**) or (**3**) of the same subsection [the superscript in the K''s above refer to the embeddings (**1**), (**2**) or (**3**)].

[§] I follow the same notation as in the examination of the symmetric cosets.

Making an analysis along the lines presented earlier in the case of symmetric cosets, I determine the particle content of the two models, which is presented in table A.4. In both cases the gauge symmetry of the four-dimensional theory cannot be broken further due to the absence of scalars.

4.2.2 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = Sp(4)/(SU(2) \times U(1))_{nonmax}$. (Case 3b′)

I consider Weyl fermions belonging in the adjoint of $G = E_8$ and the decomposition (4.2). In order the R to be embedded in E_8 as in eq. (2.61), I identify it with the $SU(2) \times U^I(1)$ appearing in the decomposition (4.2). Then, if only the CSDR mechanism was applied, the resulting gauge group would be

$$H = C_{E_8}(SU^a(2) \times U^I(1)) = E_6 \left(\times U^I(1) \right). \tag{4.18}$$

Note that again appears an enhancement of the gauge group. Similarly with previously discussed cases, the additional $U(1)$ factor in the parenthesis corresponds only to a global symmetry. The observation (4.18) suggests that we could have considered the following embedding of $R = SU(2) \times U(1)$ into E_8[¶],

$$\begin{aligned} E_8 \; &\supset SU'(3) \times E_6 \supset SU^a(2) \times U^{I'}(1) \times E_6 \\ \mathbf{248} &= (\mathbf{1},\mathbf{1})_{(0)} + (\mathbf{1},\mathbf{78})_{(0)} + (\mathbf{3},\mathbf{1})_{(0)} + (\mathbf{2},\mathbf{1})_{(-3)} + (\mathbf{2},\mathbf{1})_{(3)} \\ &\quad + (\mathbf{1},\mathbf{27})_{(2)} + (\mathbf{1},\overline{\mathbf{27}})_{(-2)} + (\mathbf{2},\mathbf{27})_{(-1)} + (\mathbf{2},\overline{\mathbf{27}})_{(1)} \,. \end{aligned} \tag{4.19}$$

The $R = SU(2) \times U^I(1)$ content of vector and spinor of $B_0 = S/R = Sp(4)/(SU(2) \times U^I(1))_{non-max}$ can be read in the last two columns of table 2.2 (case '$\mathbf{b}'$'). According to the CSDR rules then, the surviving scalars in four dimensions would transform as $\mathbf{27}_{(-2)}$, $\mathbf{27}_{(1)}$, $\overline{\mathbf{27}}_{(2)}$ and $\overline{\mathbf{27}}_{(-1)}$ under $H = E_6(\times U^I(1))$. The four-dimensional theory would also contain fermions belonging in $\mathbf{78}_{(0)}$ of H (gaugini of the model), two copies of left-handed fermions belonging in $\mathbf{27}_{L(2)}$ and $\mathbf{27}_{L(-1)}$ and one fermion singlet transforming as $\mathbf{1}_{(0)}$ under the same gauge group.

The freely acting discrete symmetries, $F^{S/R}$, of the coset space $Sp(4)/(SU(2) \times U(1))_{non-max}$, are the center of S, Z(S) $= \mathbb{Z}_2$ and the Weyl, W $= \mathbb{Z}_2$. Then, employing the W discrete symmetry, I find that the resulting four-dimensional gauge group is either

$$(i) \qquad K'^{(\mathbf{1})} = C_H(T^H) = SO(10) \times U(1) \left(\times U^I(1) \right), \qquad \text{or} \tag{4.20}$$

$$(ii) \qquad K'^{(\mathbf{2,3})} = C_H(T^H) = SU(2) \times SU(6) \left(\times U^I(1) \right), \tag{4.21}$$

depending on the embedding of $\mathbb{Z}_2 \hookrightarrow E_6$ I choose to consider (see subsection 3.3.1).

[¶]This decomposition is in accordance with the Slansky tables but with opposite $U(1)$ charge.

On the other hand, if I employ the W $\times$ Z(S) $= \mathbb{Z}_2 \times \mathbb{Z}_2$ combination of discrete symmetries, the resulting four-dimensional gauge group is either

$$(iii) \qquad K'^{(\mathbf{2}')} = C_H(T^H) = SU(2) \times SU(6) \Big(\times U^I(1) \Big) , \tag{4.22}$$

in case I embed the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ into the E_6 gauge group as in the embedding $(\mathbf{2}')$ of subsection 3.3.2, or

$$(iv) \qquad K'^{(\mathbf{3}')} = C_H(T^H) = SU^{(i)}(2) \times SU^{(ii)}(2) \times SU(4) \times U(1) \Big(\times U^I(1) \Big) , \tag{4.23}$$

in case I choose to embed the discrete symmetry as in the embedding $(\mathbf{3}')$ of the same subsection.

Making a similar analysis as before, I determine the particle content of the four different models, which is presented in table A.4. In all cases the gauge symmetry of the resulting four-dimensional theory cannot be broken further by a Higgs mechanism due to the absence of scalars.

4.2.3 Reduction of $G = E_8$ over $B = B_0/F^{B_0}$, $B_0 = S/R = SU(3)/(U^(1) \times U(1))$. (Case 4c$'$)

I consider Weyl fermions in the adjoint of $G = E_8$ and the decomposition (4.8). In order the $R = U(1) \times U(1)$, to be embedded in E_8 as in eq. (2.61) one has to identify it with the $U^I(1) \times U^{II}(1)$ appearing in the decomposition (4.8). Then, if only the CSDR mechanism was applied, the resulting four-dimensional gauge group would be

$$H = C_{E_8}(U^I(1) \times U^{II}(1)) = E_6 \Big(\times U^I(1) \times U^{II}(1) \Big) . \tag{4.24}$$

Note again that an enhancement of the gauge group appears, whereas the additional $U(1)$ factors correspond to global symmetries. The observation (4.24) suggests that I could have considered the following embedding of $R = U(1) \times U(1)$ ito E_8,

$$E_8 \supset SU(3) \times E_6 \supset (SU^a(2) \times U^{II}(1)) \times E_6 \supset E_6 \times U^I(1) \times U^{II}(1) \tag{4.25}$$

$$\begin{aligned} E_8 \ &\supset E_6 \times U^I(1) \times U^{II}(1) \\ \mathbf{248} = &\mathbf{1}_{(0,0)} + \mathbf{1}_{(0,0)} + \mathbf{78}_{(0,0)} + \mathbf{1}_{(-2,0)} + \mathbf{1}_{(2,0)} + \mathbf{1}_{(-1,3)} + \mathbf{1}_{(1,-3)} \\ &+ \mathbf{1}_{(1,3)} + \mathbf{1}_{(-1,-3)} + \mathbf{27}_{(0,-2)} + \overline{\mathbf{27}}_{(0,2)} + \mathbf{27}_{(-1,1)} + \overline{\mathbf{27}}_{(1,-1)} \\ &+ \mathbf{27}_{(1,1)} + \overline{\mathbf{27}}_{(-1,-1)} . \end{aligned} \tag{4.26}$$

The $R = U^I(1) \times U^{II}(1)$ content of vector and spinor of $B_0 = S/R = SU(3)/(U^I(1) \times U^{II}(1)$ can be read in the last two columns of table 2.2 (case '$\mathbf{c}'$'). The closure of the tensor products of the reps appeared in the (4.26) above and especially the

$$\mathbf{27}_{(0,-2)} \times \mathbf{27}_{(-1,1)} = \overline{\mathbf{27}}_{(-1,-1)} + \mathbf{351}_{a(-1,-1)} + \mathbf{351}'_{s(-1,-1)}$$

suggests the identification

$$\mathbf{27}_{(0,-2)} \leftrightarrow (a,c)\,, \qquad \mathbf{27}_{(-1,1)} \leftrightarrow (b,d)\,,$$

for the $U(1)$ charges of the R vector and spinor content, i.e. $a = 0$, $c = -2$, $b = -1$ and $d = 1$. Then, according to the CSDR rules, the four-dimensional theory would contain scalars which belong in $\mathbf{27}_{(0,-2)}$, $\mathbf{27}_{(-1,1)}$, $\mathbf{27}_{(1,1)}$, $\overline{\mathbf{27}}_{(0,2)}$, $\overline{\mathbf{27}}_{(1,-1)}$, and $\overline{\mathbf{27}}_{(-1,-1)}$ of $H = E_6(\times U^I(1) \times U^{II}(1))$. The resulting four-dimensional theory would also contain gaugini transforming as $\mathbf{78}_{(0,0)}$ under H, two copies of left-handed fermions belonging in $\mathbf{27}_{L(0,-2)}$, $\mathbf{27}_{L(-1,1)}$, $\mathbf{27}_{L(1,1)}$ and two fermion singlets belonging in $\mathbf{1}_{(0,0)}$ and $\mathbf{1}_{(0,0)}$ of the same gauge group.

The freely acting discrete symmetries, $F^{S/R}$, of the coset space $SU(3)/(U(1) \times U(1))$ (case '**c′**' in table 2.2), are the center of S, Z(S) $= \mathbb{Z}_3$ and the Weyl, W $= \mathbf{S}_3$. Then according to the list (3.4) only the $\mathbb{Z}_2 \subset$ W discrete symmetry is an interesting case to be examined further.

Then, employing the $\mathbb{Z}_2$ subgroup of the W $= \mathbf{S}_3$ discrete symmetry leads to a four-dimensional theory either with gauge group

$$(i) \qquad K'^{(\mathbf{1})} = C_H(T^H) = SO(10) \times U(1) \Big(\times U^I(1) \times U^{II}(1) \Big)\,, \qquad \text{or} \tag{4.27}$$

$$(ii) \qquad K' = C_H(T^H) = SU(2) \times SU(6) \Big(\times U^I(1) \times U^{II}(1) \Big) \tag{4.28}$$

depending on the embedding of $\mathbb{Z}_2 \hookrightarrow E_6$ I choose to consider (see sec. 3.3.1).

Making a similar analysis as before, I determine the particle content of the two models as follows.

Case (i). The resulting four-dimensional theory contains gaugini which transform as

$$\mathbf{1}_{(0,0,0)}\,, \qquad \mathbf{45}_{(0,0,0)}$$

under $K'^{(\mathbf{1})}$, a set of fermion singlets which belong in

$$\mathbf{1}_{(0,0,0)}\,, \qquad \mathbf{1}_{(0,0,0)}\,,$$

of $K'^{(\mathbf{1})}$ and a set of chiral fermions which belong in one of the linear combinations

$$\left\{ \begin{array}{l} \mathbf{1}_{L(-4,0,-2)} + \mathbf{1}'_{L(-4,0,-2)}\,, \\ \mathbf{10}_{L(-2,0,-2)} + \mathbf{10}'_{L(-2,0,-2)}\,, \\ \mathbf{16}_{L(1,0,-2)} - \mathbf{16}'_{L(1,0,-2)}\,, \end{array} \right\}, \qquad \left\{ \begin{array}{l} \mathbf{1}_{L(-4,-1,1)} + \mathbf{1}'_{L(-4,-1,1)}\,, \\ \mathbf{10}_{L(-2,-1,1)} + \mathbf{10}'_{L(-2,-1,1)}\,, \\ \mathbf{16}_{L(1,-1,1)} - \mathbf{16}'_{L(1,-1,1)}\,, \end{array} \right\},$$

or

$$\left\{ \begin{array}{l} \mathbf{1}_{L(-4,1,1)} + \mathbf{1}'_{L(-4,1,1)}\,, \\ \mathbf{10}_{L(-2,1,1)} + \mathbf{10}'_{L(-2,1,1)}\,, \\ \mathbf{16}_{L(1,1,1)} - \mathbf{16}'_{L(1,1,1)} \end{array} \right\}$$

of the same gauge group, depending on the $\mathbb{Z}_2$ subgroup of $\mathbf{S}_3$ that I choose to consider (see table 2.2).

Case (ii). The resulting four-dimensional theory contains gaugini which transform as

$$\mathbf{1}_{(0,0)}\,, \qquad \mathbf{1}_{(0,0)}\,, \qquad (\mathbf{3},\mathbf{1})_{(0,0)}\,, \qquad (\mathbf{1},\mathbf{35})_{(0,0)}$$

under $K'^{(\mathbf{2},\mathbf{3})}$, a set of fermion singlets which belong in

$$\mathbf{1}_{(0,0)}\,, \qquad \mathbf{1}_{(0,0)}\,,$$

of $K'^{(\mathbf{2},\mathbf{3})}$ and a set of chiral fermions which belong in one of the linear combinations

$$\left\{\begin{array}{l} (\mathbf{1},\mathbf{15})_{L(0,-2)} + (\mathbf{1},\mathbf{15})'_{L(0,-2)}\,, \\ (\mathbf{2},\overline{\mathbf{6}})_{L(0,-2)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(0,-2)}\,, \end{array}\right\}, \qquad \left\{\begin{array}{l} (\mathbf{1},\mathbf{15})_{L(-1,1)} + (\mathbf{1},\mathbf{15})'_{L(-1,1)}\,, \\ (\mathbf{2},\overline{\mathbf{6}})_{L(-1,1)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(-1,1)}\,, \end{array}\right\},$$

or

$$\left\{\begin{array}{l} (\mathbf{1},\mathbf{15})_{L(1,1)} + (\mathbf{1},\mathbf{15})'_{L(1,1)}\,, \\ (\mathbf{2},\overline{\mathbf{6}})_{L(1,1)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(1,1)}\,, \end{array}\right\}$$

of the same gauge group, depending on the $\mathbb{Z}_2$ subgroup of $\mathbf{S}_3$ that I choose to consider (see table 2.2).

Note that in both cases the gauge symmetry of the four-dimensional theory cannot be broken further by a Higgs mechanism due to the absence of scalars.

Finally, if I have used either the symmetric group of 3 permutations, $\mathbf{S}_3$, or its subgroup $\mathbb{Z}_3 \subset \mathbf{S}_3$, I could not form linear combinations among the two copies of the CSDR-surviving left-handed fermions and no fermions would survive in four dimensions.

4.3 Discussion

The CSDR is a consistent dimensional reduction scheme [26, 58–61], as well as an elegant framework to incorporate in a unified manner the gauge and the ad-hoc Higgs sector of spontaneously broken four-dimensional gauge theories using the extra dimensions. The kinetic terms of fermions were easily included in the same unified description. A striking feature of the scheme concerning fermions was the discovery that chiral ones can be introduced [27] and moreover they could result even from vector-like reps of the higher dimensional gauge theory [13, 18]. This possibility is due to the presence of non-trivial background gauge configurations required by the CSDR principle, in accordance with the index theorem. Another striking feature of the theory is the possibility that the softly broken sector of the four-dimensional supersymmetric theories can result from a higher-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory with only a vector supermultiplet, when is dimensionally reduced over non-symmetric coset spaces [28–30, 62]. Another interesting feature useful in realistic model searches is the possibility to deform the metric in certain non-symmetric coset spaces and introduce more than one scales [18, 52, 53].

Recently there exist a revival of interest in the study of compactifications with internal manifolds six-dimensional non-symmetric coset spaces possessing an $SU(3)$-structure

within the framework of flux compactifications [36, 37, 63–70]. In this framework the CSDR of the heterotic ten-dimensional gauge theory is an extremely interesting problem. Here, starting with a supersymmetric $\mathcal{N} = 1$, E_8 gauge theory in ten dimensions I made a complete classification of the models obtained in four dimensions after reducing the theory over all multiply connected six-dimensional coset spaces, resulting by moding out all the freely acting discrete symmetries on these manifolds, and using the Wilson flux breaking mechanism in an exhaustive way. The results of the extended investigation have been presented in [71, 72]. Despite some partial success, my result is that the two mechanisms used to break the gauge symmetry, i.e. the geometric breaking of the CSDR and the topological of the Hosotani mechanism are not enough to lead the four-dimensional theory to the SM or some interesting extension as the MSSM. However, in the old CSDR framework one can think of some new sources of gauge symmetry breaking, such as new scalars coming from a gauge theory defined even in higher dimensions [73,74]. Much more interesting is to extend my examination in a future study of the full ten-dimensional $E_8 \times E_8$ gauge theory of the heterotic string. Moreover in that case one does not have to be restricted in the study of freely acting discrete symmetries of the coset spaces and can extent the analysis including orbifolds [75–83]. More possibilities are offered in refs [84–87].

Chapter 5

Noncommutative generalisations and Motivation

Main disadvantage of the CSDR scheme when applied over continuum homogeneous coset spaces is the introduction of divergences in the quantum theory when higher modes in the Kaluza-Klein expansion are kept [9]. However, assuming the extra dimensions to form continuum and homogeneous spaces is not the only possibility. Noncommutative modifications of the extra dimensions have been proposed some years ago [88]. Their phenomenological consequences [89] and their connection with string theory [90,91] have been also studied. In this more modest version of noncommutative generalisation one provides the extra dimensions with noncommutative characteristics while keeping the continuum nature of the ordinary four-dimensional Minkowski space. Special case of noncommutative spaces are these which are approximated by finite complex matrix algebras which are defined over them and are known as '*fuzzy spaces*' in the literature [92]. One possibility of dimensional reduction over noncommutative internal spaces of this kind is to generalise appropriately the CSDR scheme (Fuzzy-CSDR) [93].

In chapter 6, I present the necessary mathematical framework and ideas to describe noncommutative generalisations of the ordinary Kaluza-Klein theories and their dimensional reduction.

In chapter 7, I review and further explain the Fuzzy-CSDR scheme. Important conclusions of the study are (i) the enhancement of the gauge symmetry due to the noncommutative characteristics of the internal manifold and (ii) the fact that dimensional reduction of gauge theories over such spaces lead to renormalizable theories in four dimensions.

In chapter 8, motivated by the interesting features of the Fuzzy-CSDR scheme, I examine the inverse problem, i.e. whether obtaining fuzzy extra dimensions as a vacuum solution of a four-dimensional but renormalizable potential is possible. Indeed, starting from the most general renormalizable potential in four dimensions, fuzzy extra dimensions are dynamically generated [94]. Furthermore the initial gauge symmetry found to break spontaneously towards phenomenologically interesting patterns.

In chapter 9, I assume that in the the energy regime where the noncommutativity is expected to be valid, the ordinary spacetime cannot be described as a continuum and investigate possible generalisations of Einstein gravity. Elementary cells of length scale μ_P^{-1} (with μ_P denoting the Planck mass) are expected to form out. Then the ordinary spacetime is just a limit of this noncommutative 'phase'. The purpose of the study of noncommutative spacetime is, firstly to explore new mathematical ideas and generalisations of the Einstein gravity. Secondly, having assumed the extra dimensions in the Planck mass regime to appear noncommutative characteristics would be no profound reason for this behaviour not to be the case in the ordinary dimensions too.

Finally, in chapter 10 I summarise the conclusions of the present research work.

Chapter 6

Noncommutative modifications of Kaluza-Klein theory

Here, I present the necessary mathematical framework and ideas to describe noncommutative modifications of Kaluza-Klein theories and their subsequent dimensional reduction. Furthermore, I choose to assume a modest version of noncommutative generalisation, providing the extra dimensions with noncommutative characteristics while keeping the continuum nature of the ordinary four-dimensional Minkowski space. Following [92], I describe the noncommutative nature of the extra space by a finite complex matrix algebra, M_N, defined over it. This promote, the Kaluza-Klein theory, based on a manifold $M^4 \times B$ and described by the $\mathcal{C} \otimes \mathcal{C}(B)$ tensor product of associated algebras of continuum functions to a theory with a geometry described by the algebra $\mathcal{A} = \mathcal{C} \otimes M_N$. The motivation for such generalisation is the lack of a well-defined notion of the point, embedded in its very nature. It is the same characteristic of the quantum mechanical phase space, expressed for the first time by the Heisenberg uncertainty relations. The extra space coordinates, then, become noncommutative operators and by analogy with quantum mechanics, points are expected to be replaced by elementary cells. It is this cellular structure which serve as an ultraviolet cut-off similar to a lattice structure. As a result, divergences coming from the extra compactification space are no longer appear in quantum theory, making this noncommutative modification of Kaluza-Klein theories an appealing configuration for further investigation.

6.1 Introductory remarks and motivation

The simplest definition of the noncommutative geometry is that is a geometry in which the coordinates does not commute. The most familiar example of such a space is the quantised version of a two-dimensional phase space. Basic characteristic of this example is the built-in uncertainty in the simultaneous measurements of its coordinates. The notion of a point is no longer well-defined; this is realised by the well-known Heisenberg

uncertainty relations. The lack of a definition of a point constitutes an essential built-in characteristic of noncommutative geometries, too. Noncommutative geometries are in this sense 'pointless geometries'. The mathematical formulation of these kind of geometries has been studied in a series of works and a variety of different approaches have been developed by now. For reviews and references therein consult [92, 95–98]. Among them and following the example of quantum mechanics, one promotes the ordinary coordinates of space and time to noncommutative operators. Then the notion of a pure state replaces that of the point and derivations of the algebra replace vector fields [92, 99]. Then the notion of points is expected to be replaced by the notion of the elementary cells; each of them correspond to the minimal measurable 'area' in the noncommutative 'coordinate' space. Noncommutative geometries, in this sense realised, are close to the example of lattice gauge theories. These were developed for the study of the non-perturbative regime of various physical systems, a goal which is achieved by an *ad-hoc* discretisation of the configuration (and internal) space of the original theory [100]. However, in lattice approach some of the continuous transformations of the original theory are not preserved by this very discretisation procedure. This is not always the case with the noncommutative geometries. The transformation properties of the original theory could be preserved by the 'quantisation' of the commutative space, and in this sense, noncommutative geometry could be a better description of nonperturbative effects at least at Planck length-scale. In this regime there is theoretical circumstantial evidence that an elementary length may exist. Therefore, a quantum-mechanical description of the space may be appropriate.

On the other hand, the question of existence of hidden extra dimensions and the actual geometry of their compactification space (if any), is still an open problem. One of the first negative answers was given by Kaluza [7] and Klein [8] in their attempt to introduce extra dimensions in order to unify the gravitational field with electromagnetism. Later, with the advent of more elaborate gauge fields, it was proposed that this internal space could be taken as a compact Lie group or something more general as a coset space. Dimensional reduction over such internal space resulted to non-abelian gauge groups in four dimensions [10]. In the previous chapters, I reviewed such ideas and I studied in detail the CSDR case. However, the great disadvantage of using homogeneous coset spaces as the extra dimensions is that they introduce even more divergences in the quantum theory and lead to an infinite spectrum of new particles. In fact the structure is strongly redundant and most of it has to be discarded by a truncation procedure of a possible dimensional reduction. If on the other hand one use matrix approximations of coset spaces (fuzzy coset spaces) for the extra dimensions, which are spaces of a special noncommutative structure*, the theory turns out to be power counting renormalizable; the fuzzy spaces are approximated by matrices of finite dimensions and only a finite number of counterterms are required to make the Lagrangian renormalizable.

In section 6.2, I give a short formulation of the Kaluza-Klein theory in a suitable language for its noncommutative generalisation. In section 6.3, I promote the extra compactifi-

*Each fuzzy coset space is described by an appropriate Lie algebra. I discuss the case of the two-dimensional fuzzy sphere as an example in section 6.5.

cation space of Kaluza-Klein theory to its noncommutative approximation. I assume that the internal structure of the theory is described by a noncommutative geometry in which the notion of a point does not exist. As a particular example of this noncommutative generalisation, I shall choose only internal structures which give rise to a finite spectrum of particles, namely noncommutative spaces described by finite complex matrix algebras [92]. In section 6.4, I explain, following [88, 101], how this idea can lead to a noncommutative modification of Kaluza-Klein theory. Finally, in section 6.5, I give an extended description of the fuzzy sphere 'manifold', S^2_N, and the differential geometry which is defined over it. I stress - among others - the appearance of elementary cells (elementary measurable 'area') and the preservation of the continuum symmetries of the ordinary sphere. These are built-in characteristics of the matrix approximated manifolds.

6.2 Kaluza-Klein theory

In its local aspects Kaluza-Klein theory is described by an extended space-time $V \equiv M^D = M^4 \times B$, of dimension $D = 4 + d$ and with coordinates $x^M = (x^\mu, x^a)$. The x^μ are the coordinates of space-time which I consider here to be Minkowski space; the x^a are the coordinates of the internal space, which in this section will be implicitly supposed to be space-like and 'small'. In section 6.4 it will be of purely algebraic nature and not necessarily 'small'.

Let $\mathcal{C}(V)$ be the commutative and associative algebra of smooth complex-valued functions on V. I define the sum and product of two functions by the sum and product of the value of the function at each point. The commutative, associative and distributive rules follow then from those of $\mathbb{R}$. It is evident that every such V space can be embedded in an Euclidean space of sufficiently high dimension $D' > D$. It is defined then by a set of $D' - D$ relations in the Euclidean coordinates x^M; the algebra $\mathcal{C}(V)$ can be considered as a quotient of the algebra of smooth function over $\mathbb{R}^{D'}$ by the ideal generated by these relations. I recall that an *ideal* of an algebra is a subalgebra which is stable under multiplication by a general element of the algebra. The algebra $\mathcal{C}(V)$ has many ideals, for example the subalgebra of functions which vanish on any close set in V. The algebra M_N of $N \times N$ complex matrices, on the other hand has no proper ideals.

Let now X be a smooth vector field on the V manifold considered above. I denote the vector space of all such X as $\mathcal{X}(V)$ which is a left $\mathcal{C}(V)$ module; if $f \in \mathcal{C}(V)$ and $X \in \mathcal{X}(V)$ then $fX \in \mathcal{X}(V)$. We recall that a *left (right) module* is a vector space on which there is a left (right) action of the algebra. Let ∂_M be the natural basis of the vectors on the embedding space $\mathbb{R}^{D'}$. It can be proved that any X can be written as a linear combination $X = X^M \partial_M$ with $X^M \in \mathcal{C}(V)$. As a $\mathcal{C}(V)$-module, $\mathcal{X}(V)$ is finitely generated although not uniquely determined. In the case that V space is a *parallelizable manifold*, the derivations $\{e_M\}$ defined over $\mathcal{X}(V)$ are linearly independent and the linear combination $X = X^M e_M$, $M = 1, \ldots, D$, unique. The set $\{e_M\}$ forms the basis for the globally defined moving frame over the V. An embedding space larger than $\mathbb{R}^D$ is no

longer required and the set of derivations $\{e_M\}$ happens to coincide with the $\{\partial_M\}$ basis noted above. We note however, that in general a moving frame can be defined only locally on V.

I recall that the definition of the *Lie bracket* $[X, Y]$ of two vector fields is

$$[X, Y]f = (XY - YX)f = (X^M \partial_M Y^N - Y^M \partial_M X^N)\partial_N f\,,$$

where f an arbitrary element of $\mathcal{C}(V)$. Note that the Lie bracket of two vector fields is also a vector field. For the particular case of the parallelizable manifolds I can write the Lie bracket as

$$[e_M, e_N] = C^K{}_{MN} e_K\,, \tag{6.1}$$

where the structure constants $C^K{}_{MN}$ are elements of $\mathcal{C}(V)$.

Any vector field can be defined as *derivation* of the algebra $\mathcal{C}(V)$, if there is a linear map of $\mathcal{C}(V)$ onto itself which satisfies the *Leibniz rule*

$$X(f\ g) = (X\ f)g + f(X\ g)\,. \tag{6.2}$$

Then, I can identify the vector space $\mathcal{X}(V)$ with the space of derivations

$$\mathcal{X}(V) \equiv \mathrm{Der}(\mathcal{C}(V))\,. \tag{6.3}$$

I choose the $\mathcal{C}(V)$ to be an algebra of complex-valued functions. Using complex conjugation a $*$-operation: $f \mapsto f^*$ can be defined. The algebra is assumed to close under this operation being a so-called $*$-*algebra*. I also assume that the $\mathcal{X}(V)$ elements satisfy the reality condition

$$(Xf)^* = X(f^*), \quad \forall\ X \in \mathcal{X}(V) \text{ and } \forall\ f \in \mathcal{C}(V)\,. \tag{6.4}$$

It is straightforward to check that if the f elements of $*$-algebra were hermitian, the $*$-algebra would fail to close under noncommutative multiplication; $(fg)^* = g^* f^*$ which is not equal with $f^* g^*$. Therefore, although I am interested in real manifolds and their noncommutative counterparts, algebras of complex functions are necessary to be considered.

A *differential form* of order p or p-*form* α is a p-linear completely antisymmetric map of the vector space $\mathcal{X}(V)$ into $\mathcal{C}(V)$. In particular if $f \in \mathcal{C}(V)$ and $X_1, \ldots, X_p$ are p-vector fields then

$$(\alpha f)(X_1, \cdots, X_p) = (f\alpha)(X_1, \cdots, X_p) = f(\alpha(X_1, \cdots, X_p))\,,$$

that is the value of $\alpha(X_1, \ldots, X_p)$ at a point of V depends only on the values of the vector fields at that point. The set $\Omega^p(V)$ of p-forms is a $\mathcal{C}(V)$-module.

The *exterior product* $\alpha \wedge \beta$ of $\alpha \in \Omega^p(V)$ and $\beta \in \Omega^q(V)$, on the other hand, is an $\Omega^{p+q}(V)$ element defined by

$$\alpha \wedge \beta(X_1, \cdots, X_{p+q}) = \frac{1}{(p+q)!} \sum \epsilon(i, j)\alpha(X_{i_1}, \cdots, X_{i_p})\beta(X_{j_1}, \cdots, X_{j_q})\,, \tag{6.5}$$

where the summation is taken over all the possible partitions of $(1, \ldots, p+q)$ into $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_q)$ and $\epsilon(i,j)$ is the signature of the corresponding permutation. It is graded commutative, $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$, because the algebra $\mathcal{C}(V)$ is commutative. This is not generally the case in the noncommutative algebras and I shall usually write the exterior product of two forms α and β simply as $\alpha\beta \equiv \alpha \wedge \beta$. I also define $\Omega^0(V) = \mathcal{C}(V)$ and the $\Omega^*(V)$ as the set of all $\Omega^p(V)$, $p = 1, \ldots, D$. In the particular case of the parallelizable manifolds

$$\Omega^*(V) = \mathcal{C}(V) \otimes \Lambda^* \,, \tag{6.6}$$

where Λ^* is the exterior algebra over the complex numbers generated by the frame.

The *exterior derivative* $d\alpha$ of $\alpha \in \Omega^p(V)$ is defined by the formula

$$\begin{aligned} d\alpha(X_0, \cdots, X_p) = & \frac{1}{p+1} \sum_{i=0}^{p} (-1)^i X_i(\alpha(X_0, \cdots, \hat{X}_i, \cdots, X_p)) \\ & + \frac{1}{p+1} \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p) \,, \end{aligned} \tag{6.7}$$

where the hat symbol means that the symbol underneath is omitted. The case $p = 0$ is especially interesting

$$df(X) = X\, f \tag{6.8}$$

leading to the $df(\partial_M) = \partial_M f$ if the linear expansion $X = X^M \partial_M$ of a vector field is taken into account. This relation has the same content with the $df = (\partial_M f) dx^M$ definition of ordinary calculus differential. One passes from one to the other by using the particular case $dx^M(\partial_N) = \delta^M{}_N$. Namely, by the choice of $\{dx^M\}$ basis to be dual to the $\{\partial_M\}$ one. The derivations form a vector space (the tangent space) and (6.8) defines the df as an element of the dual space (contangent space).

Let now $\alpha \in \Omega^p(V)$ and $\beta \in \Omega^*(V)$. Then d satisfies the condition

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \,, \tag{6.9}$$

which obviously does not satisfy the Leibniz rule. It is a graded derivation of $\Omega^*(V)$. From (6.7) it follows that

$$d^2 = 0 \tag{6.10}$$

and decomposes the set $\Omega^*(V)$ of all forms into a direct sum

$$\Omega^*(V) = \Omega^{*+}(V) \oplus \Omega^{*-}(V) \tag{6.11}$$

of even and odd forms respectively. The differential d takes one into another.

The couple $(\Omega^*(\mathcal{C}(V), d)$ is called a graded differential algebra or a differential calculus over $\mathcal{C}(V)$. I shall show later that $\mathcal{C}(V)$ need not be commutative and $\Omega^*(\mathcal{C}(V))$ need

not be graded commutative. Over each algebra $\mathcal{C}(V)$, be it commutative or not, there can exist a multitude of differential calculi.

However there are two additional elements of the differential calculus which are important for its noncommutative generalisation and have not been discussed so far. Firstly, I note that the *Lie derivative* of vector field Y with respect to the vector field Y, to be $L_X Y = [X, Y]$. Indeed, defining a smooth map ϕ from V to V', it induces a map

$$\mathcal{C}(V') \xrightarrow{\phi^*} \mathcal{C}(V) \,, \qquad \phi^* f = f \circ \phi^*$$

which has a natural extension to the set of all forms

$$\Omega^*(V') \xrightarrow{\phi^*} \Omega^*(V) \,, \qquad \phi^*(df) = d(\phi^* f) \,.$$

If ϕ is a diffeomorphism one can identify V' and V and consider ϕ^* as an automorphism of the $\mathcal{C}(V)$ and $\Omega^*(V)$. In this case ϕ also induce a map ϕ_* of $\mathcal{X}(V)$ onto itself by the formula

$$(\phi_* X) f = (\phi^{*-1} X \phi^*) f, \qquad f \in \mathcal{C}(V) \,.$$

Letting ϕ^* be a local one-parameter group of diffeomorphisms of V generated by a vector field X, $\phi_t^* f = f + tXf + \mathcal{O}(t^2)$, then acting on a vector field Y

$$\phi_{t*} Y = Y - t \; L_X Y + \mathcal{O}(t^2)$$

with the Lie derivative of Y be calculated as $L_X Y = [X, Y]$. By requiring that it be a derivation, the Lie derivative can be extended to a general element of the tensor algebra over $\mathcal{X}(V)$. On the other hand, the Lie derivative of an arbitrary function $f \in \mathcal{C}(V)$ is given by $L_X f = Xf$ and the Lie derivative of a $\alpha \in \Omega^p(V)$ is given by the formula

$$(L_X \alpha)(Y_1, \ldots, Y_p) = X\alpha(Y_1, \ldots, Y_p) - \sum_1^p \alpha(Y_1, \ldots, [X, Y_i], \ldots, Y_p) \ldots) \,. \tag{6.12}$$

Again, by requiring that it be a derivation, the Lie derivative can be extended to a general element of the tensor algebra over $\Omega^1(V)$.

Secondly, the *interior product* i_X is defined to be the map of $\Omega^{p+1}(V)$ into $\Omega^p(V)$ and is given for $\alpha \in \Omega^{p+1}(V)$ by the equation

$$(i_X \alpha)(X_1, \cdots, X_p) = (p+1)\alpha(X, X_1, \cdots, X_p) \,. \tag{6.13}$$

It is actually a graded derivation of $\Omega^*(V)$. By setting $i_X f = 0$, one easily derives the following formula

$$L_X = i_X d + d\, i_X \,, \tag{6.14}$$

which relates the Lie derivative with the differential and the interior product.

To form tensors one must be able to define tensor products, for example the tensor product $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ of $\Omega^1(V)$ with itself. The $\mathcal{C}(V)$ subscript is to denote the fact that I identify $\xi f \otimes \eta$ with $\xi \otimes f\eta$ for every element f of the $\mathcal{C}(V)$ algebra. This is important in the applications of section 6.4. It means also that one must be able to multiply the elements of $\Omega^1(V)$ on the left and on the right by the elements of the algebra $\mathcal{C}(V)$. If $\mathcal{C}(V)$ is commutative of course these two operations are equivalent. When $\mathcal{C}(V)$ is an algebra of functions this left/right linearity is equivalent to the property of locality. It means that the product of a function with a one-form at a point is again a one-form at the same point. This property distinguishes the ordinary product from other, non-local, products such as the convolution. In the noncommutative case there are no points and locality can not be defined; it is replaced by the property of left and/or right linearity with respect to the algebra.

To define a metric and covariant derivatives on the extended space-time I set $\theta^M = dx^M$ in the absence of a gravitational field, that are dual to the basis of derivations, according to our previous discussion. Then, I have $d\theta^M = 0$. The extended Minkowski metric can be defined as the map

$$g(\theta^M \otimes \theta^N) = g^{MN} \tag{6.15}$$

which associates to each element $\theta^M \otimes \theta^N$ of the tensor product $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ the contravariant components g^{MN} of the (extended) Minkowski metric. There are of course several other definitions of a metric which are equivalent in the case of ordinary geometry but the one I have given has the advantage of an easy extension to the noncommutative case. The map g must be bilinear so that I can define for arbitrary one-forms $\xi = \xi_M \theta^M$ and $\eta = \eta_M \theta^M$

$$g(\xi \otimes \eta) = \xi_M \eta_N g(\theta^M \otimes \theta^N) = \xi_M \eta_N g^{MN}\,. \tag{6.16}$$

In that case the elementary line element defined over the manifold can be rescaled to

$$ds^2 = \eta_{\mu\nu}\theta^\mu\theta^\nu + g_{ab}\theta^a\theta^b\,, \tag{6.17}$$
$$\eta_{\mu\nu} = diag(-1,1,1,1)\,, \qquad \text{and} \qquad g_{ab} = diag(1,\dots,1)$$

which define a local comoving frame. This is also possible in the noncommutative case but for a frame $\{\theta^a\}$, the basis of which, commutes with the noncommutative coordinates of the 'manifold'.

I introduce a gauge potential by first defining a covariant derivative. Let ψ be a complex-valued function which I shall consider as a 'spinor field' with no Dirac structure and let $\mathcal{H}$ be the space of such 'spinor fields'. A covariant derivative is a rule which associates to each such ψ in $\mathcal{H}$ a spinor-one-form $D\psi$. It is a map

$$\mathcal{H} \xrightarrow{D} \Omega^1(V) \otimes_{\mathcal{C}(V)} \mathcal{H} \tag{6.18}$$

from $\mathcal{H}$ into the tensor product $\Omega^1(V) \otimes_{\mathcal{C}(V)} \mathcal{H}$. In the absence of any topological complications the function $\psi = 1$ is a spinor field and I can define a covariant derivative by the rule

$$D(1) = A \otimes 1. \tag{6.19}$$

The (local) gauge transformations are the complex-valued functions with unit norm and so A must be a one-form with values in the Lie algebra of the unitary group $U(1)$, that is, the imaginary numbers. An arbitrary spinor field ψ can always be written in the form $\psi = f \cdot 1 = 1 \cdot f$ where f is an element of the algebra $\mathcal{C}(V)$. The extension to ψ of the covariant derivative is given by the Leibniz rule

$$D\psi = df \otimes 1 + A \otimes f = d\psi \otimes 1 + A \otimes \psi\,, \tag{6.20}$$

an equation which I simply write in the familiar form $D\psi = d\psi + A\,\psi$. Using the graded Leibniz rule one have

$$D(\alpha\psi) = d\alpha \otimes \psi + (-1)^p\,\alpha\,D\psi\,, \tag{6.21}$$

the covariant derivative can be extended to higher-order forms and the field strength F defined by the equation

$$D^2\psi = F\psi. \tag{6.22}$$

To introduce the gravitational field it is always possible to maintain (6.15) but at the cost of abandoning $d\theta^M = 0$. This is known as the moving-frame formalism [92]. In the presence of gravity the dx^M become arbitrary one-forms θ^M. The differential df can be written

$$df = (e_M f)\theta^M \tag{6.23}$$

which is still of the form $df = (\partial_M f)dx^M$ provided one introduces modified derivations e_M. An equation

$$df(e_M) = e_M f \tag{6.24}$$

equivalent to $df(\partial_M) = \partial_M f$ can be written if one imposes the relations

$$\theta^M(e_N) = \delta^M{}_N\,. \tag{6.25}$$

The θ^M are a (local) basis of the one-forms dual to the derivations e_M exactly as the dx^M are dual to the ∂_M. Equation $d\theta^M = 0$ must be replaced by the structure equations

$$d\theta^M = -\frac{1}{2}C^M{}_{NK}\theta^N\theta^K \tag{6.26}$$

which express simply the fact that the differential of a one-form is a 2-form and can be thus written out in terms of the (local) basis $\theta^M\theta^N$. The structure equations can normally not be written globally. In the noncommutative case such equations do not in general make sense because the differential forms need not have a basis.

The covariant derivative can now be defined as follows. For the considered (local orthonormal) moving frame, $\{e_M\}$, with $M = 1, \ldots, D$, $D = d + 4$ and a given vector field

$X = X^M e_M$, the *covariant derivative* DX of X can be defined by the local expression $DX = (DX^M) \otimes e_M$

$$DX^M = dX^M + \omega^M{}_N X^N \, ; \tag{6.27}$$

$\omega^M{}_N$ is the so called *linear connection* a one-form in V with values in the Lie $\mathfrak{so}(3,1) \oplus \mathfrak{so}(d)$ algebra for the given compactification. The transformation properties of $\omega^M{}_N$ assure that D is a well defined map from $\mathcal{X}(V)$ to $\Omega^1(V) \otimes_{C(V)} \mathcal{X}(V)$. $\omega^M{}_N$ also has to satisfy the *structure equations*

$$T^M = d\theta^M + \omega^M{}_N \wedge \theta^N \, , \tag{6.28}$$

$$\Omega^M{}_N = d\omega^M{}_N + \omega^M{}_K \wedge \omega^K{}_N \, , \tag{6.29}$$

whereas the *torsion* T^M and the *curvature form*

$$\Omega^M{}_N = \frac{1}{2} R^M{}_{NK\Lambda} \theta^K \theta^\Lambda \tag{6.30}$$

satisfy the *Bianchi identities*

$$dT^M + \omega^M{}_N T^N = \Omega^M{}_N \wedge \theta^N \, , \tag{6.31}$$

$$d\Omega^M{}_N + \omega^M{}_K \wedge \Omega^K{}_N - \Omega^M{}_K \wedge \omega^K{}_N = 0 \, . \tag{6.32}$$

However, under the considered compactification (6.17), these relations are decomposable into four and extra dimensional part. Their decomposition and the emergent restrictions will be studied in the context of the noncommutative geometry in sections 6.3 and 6.4.

For a general introduction to Kaluza-Klein theory and references therein consult the review articles by Bailin & Love [9], Coquereaux& Jadczyk [102] or the more recent one [14].

6.3 A noncommutative geometry

According to the discussion in the previous section, the basic structure of the differential geometry of a continuum manifold can be also expressed in terms of an algebra of functions defined on the manifold. Local coordinates can be replaced by generators of the algebra whereas the vector fields by derivations. This remain valid for the case of the noncommutative spaces, which can be described by an abstract associative algebra $\mathcal{A}$ which is not necessarily commutative. Historically the most important noncommutative algebra in physics was the quantised space of non-relativistic quantum mechanics. This algebraic approach to quantum mechanics was extended and developed by Neumann [103]. Important for the developing intuition of the commutative limit of noncommutative geometry was the classical limit of quantum-mechanical systems. The symplectic geometry of quantised phase space and its relation to noncommutative geometry have been discussed for example, by [104, 105]. Spaces with noncommutative characteristics cannot called ‘manifolds’; the notion of the localisation of a point is

absent on them. This is the essential feature which makes noncommutative geometry particularly well suited for the description of the internal structure of a Kaluza-Klein theory.

On the other hand, an obvious way of a noncommutative generalisation of a classical field theory would be to replace the objects which describe the theory in the commutative case by their corresponding ones with noncommutative characteristics. The simplest suggestion is to consider the M_N algebras of finite $N \times N$ complex matrices. For establishing the various ideas of this 'Matrix geometry' and references to original works consult [92].

The motivation for introducing noncommutative geometry in Kaluza-Klein theory lies in the suggestion that space-time structure cannot be adequately described by ordinary geometry at all length scales, including those which are presumably relevant when considering hidden dimensions. There is of course no reason to believe that the extra structure can be described by the simple matrix geometries I shall consider, although this seems suggestive by the finite particle multiplets observed in nature.

To be more specific, let the M_N algebra of $N \times N$ complex matrices and $\{\lambda_a\} \in M_N$, $a = 1, \ldots, N^2 - 1$ be an antihermitean basis of the Lie algebra of the special unitary group $SU(N)$. The Killing metric is given by $g_{ab} = -\mathrm{Tr}(\lambda_a \lambda_b)$ which is going to be used for rising and lowering indices. The set $\{\lambda_a\}$ is a set of generators of M_N algebra which although not minimal is a convenient one; the derivations

$$e_a = \kappa^{-1} \mathrm{adj}(\lambda_a) \tag{6.33}$$

form a basis over the complex numbers for the derivations of M_N. They satisfy the commutation relations

$$[e_a, e_b] = m C^c{}_{ab} e_c\,, \tag{6.34}$$

where the mass scale m is defined as the inverse of the length scale κ.

Let x^μ be the coordinates of ordinary spacetime, M^4. Then the set (x^μ, λ_a) is a set of generators of the algebra $\mathcal{A} = \mathcal{C} \times M_N$. Here, I concentrate on the internal algebraic structure of the manifold. The exterior derivative of an element f of M_N, df is defined as usual by the relation (6.24). Since any element f can be written as linear combination of the derivations, $f = f^a e_a$, the relation above would mean

$$d\lambda^a(e_b) = m[\lambda_a, \lambda_b] = m C^a{}_{cb} \lambda^c\,. \tag{6.35}$$

The set of $d\lambda_a$ forms a system of generators of $\Omega^1(M_N)$ as a left or right module but is not a convenient one. Since the algebra is now noncommutative happens for example to be $\lambda_a d\lambda_b \neq d\lambda_b \lambda_a$. However because of the particular structure of M_N there is another system of generators being orthogonal to the derivations e_a

$$\theta^a(e_b) = \delta^a{}_b\,. \tag{6.36}$$

and form a dual of the derivation space basis. This set of generators is related to the $d\lambda_a$ by the equations

$$d\lambda^a = m\ C^a{}_{bc}\lambda^b\theta^c, \qquad \theta^a = \kappa\lambda_b\lambda^a d\lambda^b\,, \tag{6.37}$$

and it satisfies the same structure equations as the components of Maurer-Cartan form on the special unitary group $SU(N)$

$$d\theta^a = -\frac{1}{2}m\ C^a{}_{bc}\theta^b\theta^c\,. \tag{6.38}$$

The product in the right-hand side of this relation is the product in $\Omega(M_N)$. Although this product is not in general antisymmetric, because of the (6.36) above I have

$$\theta^b\theta^a = -\theta^a\theta^b\,. \tag{6.39}$$

The θ^a's also commute with elements of M_N and $\Omega^1(M_N)$ and can be identified by the tensor product of M_N and the dual of the vector space of derivations. The subalgebra $\Omega^*(M_N)$ generated by the θ^a is an exterior algebra. Relation (6.38) reveals that it is also a differential subalgebra, as I expected. Since I shall only use $\Omega^*(M_N)$ in what follows I shall write the product as a wedge product

$$\theta^a\theta^b = \theta^a \wedge \theta^b\,. \tag{6.40}$$

Let me choose a basis $\theta^\mu_\nu dx^\nu$ of $\Omega^1(\mathcal{C})$ over $\mathcal{C}$ and let e_μ be the Pfaffian derivations dual to θ^μ. I set, similarly with section 6.2, $M = (\mu, a)$, $1 \leq M \leq 4+(N^2-1)$, and introduce $\theta^M = (\theta^\mu, \theta^a)$ as generators of $\Omega^1(\mathcal{A})$ as left or right module and $e_M = (e_\mu, e_a)$ as a basis of derivations, $\mathrm{Der}(\mathcal{A})$, over $\mathcal{C}$. I decompose the $\Omega^1(\mathcal{A})$ in a direct sum

$$\Omega^1(\mathcal{A}) = \Omega^1_H \oplus \Omega^1_V\,, \tag{6.41}$$

of an horizontal and vertical part which are defined respectively as

$$\Omega^1_H = M_N \otimes \Omega^1(\mathcal{C})\,, \qquad \Omega^1_V = \mathcal{C} \otimes \Omega^1(M_N)\,. \tag{6.42}$$

The Ω^1_H part has basis θ^μ whereas the Ω^1_V, θ^a. One can decompose similarly the exterior derivative also

$$d = d_H + d_V\,. \tag{6.43}$$

As I have discussed in the previous section the generators θ^a of $\Omega^1(M_N)$ can be considered as a sort of moving frame. By comparing the relation (6.38) with the first structure equations for this frame,

$$d_H\theta^a = 0, \qquad d_V\theta^a + \omega^a{}_b \wedge \theta^b = T^a\,, \tag{6.44}$$

one concludes that for vanishing torsion T^a the internal structure is provided with a linear connection

$$\omega^a{}_b = -\frac{1}{2}mC^a{}_{bc}\theta^c\,, \tag{6.45}$$

being actually a curved space. Then, I have a covariant derivative D_a which differs from e_a. Alternatively, having required the linear connection $\omega^a{}_b$ to vanish, the torsion would be

$$T^a = -\frac{1}{2} m C^a{}_{bc} \theta^b \wedge \theta^c \,. \tag{6.46}$$

In this case the covariant derivative D_a and e_a would coincide in the absence of gauge couplings but the D_a would satisfy

$$[D_a, D_b] = m C^a{}_{bc} D_c \tag{6.47}$$

forming the Lie algebra of the $SU(N)$ group. In the following lines, I choose the (6.45) solution in order to avoid an extra torsion term. If α a one-form $\alpha = \alpha_a \theta^a$ then I have by definition, in the absence of torsion

$$d\alpha = (D_a \alpha_b) \theta^a \theta^b \,, \tag{6.48}$$

with

$$D_a \alpha_b = e_a \alpha_b - \frac{1}{2} m C^c{}_{ab} a_c \,. \tag{6.49}$$

Note that this covariant derivative commutes with itself when acting on elements of the algebra; the ordinary derivative does not. That is for any $f \in \mathcal{A}$,

$$D_{[a} D_{b]} f = 0 \,. \tag{6.50}$$

Note that the equations given above are with respect to an arbitrary basis λ^a but they are all tensorial in character with respect to a change of basis

$$\lambda^a \to \lambda'^a = A^a{}_b \lambda^b, \qquad (A^a{}_b) \in \mathrm{GL}(\mathrm{N}^2 - 1) \,. \tag{6.51}$$

The covariant derivative (6.49), on the other hand, transform as it should. What is unusual is that the connection transform also as a tensor and each term of (6.49) transforms as a tensor separately. This is related to the fact that on the factor M_N of the considered algebra the notion of a point is absent; therefore, there is no analog of local variation. Each θ^a corresponds to a globally defined moving frame and its transformations correspond to the set of global transformations in internal space. The change of basis (6.51) is the equivalent in M_N of a coordinate transformation in $\mathcal{C}$. Of course, I can always choose an appropriate change of basis and set the Killing metric g_{ab} equal to the Euclidean one δ_{ab}.

Finally, one can also consider the automorphisms of M_N, given by

$$\lambda^a \to \lambda'^a = g^{-1} \lambda^a g, \qquad g \in \mathrm{GL}(\mathrm{N}) \,. \tag{6.52}$$

Restricting the discussion, in favour of brevity, only in the case of infinitesimal transformations

$$\lambda^a \to \lambda'^a = \lambda^a - [f, \lambda^a], \qquad g \backsimeq 1 + f \,, \tag{6.53}$$

I see that

$$\theta'^a = \theta^a - L_X\theta^a\,, \qquad X = \mathrm{adj}(f) \tag{6.54}$$

and in general for any N-form α

$$\alpha' = \alpha - L_X\alpha\,. \tag{6.55}$$

6.4 Noncommutative Kaluza-Klein theory

Here, I describe a noncommutative modification of the Kaluza-Klein theory following the original work of [88].

Firstly, I introduce the quadratic form of signature $N^2 + 1$

$$ds^2 = g_{MN}\theta^M \otimes \theta^N = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu + g_{ab}\theta^a \otimes \theta^b\,, \tag{6.56}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. I refer to this form as a metric although it contains two terms of a slightly different nature. There is however a unique metric g_V for M_N with respect to which all the derivations e_a are Killing derivations and it is defined by $L_X g_V = 0$ if $X = X^a e_a$ be an arbitrary derivation. To within a rescaling the g_{ab} noted above are the components of g_V.

A general one-form $\theta \in \Omega^1_V$, on the other hand, can be easily constructed from the generators θ^a as

$$\theta = -m\lambda_a\theta^a\,, \tag{6.57}$$

which from relations (6.37) and (6.38) satisfies the zero-curvature condition

$$d\theta + \theta^2 = 0\,. \tag{6.58}$$

This θ turns out to be gauge invariant as I shall explain below. It satisfies with respect to the algebraic exterior derivative d_V similar conditions to the ones satisfied by the Maurer Cartan form with respect to ordinary exterior derivation of the $SU(N)$ gauge group. As I have described in section 6.3, I have a map of the trace-free elements of $f \in M_N$ onto the derivations of M_N given by $X \to X = \mathrm{adj}(f)$. The one-form θ can be defined without any reference to the θ^a as the inverse map: $\theta(X) = -f$. The one-form θ turns out to be invariant under all derivations of M_N. To within rescaling by some complex number it is the only one-form having this property.

In the commutative case a connection ω on the trivial principal $U(1)$ bundle is an antihermitean one-form which can be splitted in a horizontal part, a one-form on the base manifold and a vertical part, the Maurer-Cartan form $d\alpha$ on $U(1)$. This is

$$\omega = A + d\alpha\,. \tag{6.59}$$

The gauge potential $A \in \Omega^1(\mathcal{C})$ can be used to construct a covariant derivative on the associated vector bundle. The notion of a vector bundle can be generalised to the noncommutative case as an $\mathcal{A}$ module which is a free module of rank 1 for the case of no topological complications; it can be identified with $\mathcal{A}$ itself. This is in fact the most natural generalisation, since in the noncommutative modification of the theory which consider, the M_N has replaced the $\mathbb{C}$. Therefore, it is important to note that the $U(N)$ gauge symmetry I shall use below comes not from the rank of vector bundle but from the finite dimension of M_N matrices. The noncommutative generalisation of a gauge potential A, according to the section 6.3, is an antihermitean $\Omega^1(\mathcal{A})$ element, which can be splitted again in horizontal and vertical parts

$$\omega = A + \theta + \phi\,. \tag{6.60}$$

The A is the gauge potential belonging in the horizontal part Ω^1_H and ϕ is an element of Ω^1_V. The θ is similar to Maurer-Cartan form.

Let $\mathcal{U}$ be the unitary elements of $\mathcal{A}$. In the case I am considering here, namely $\mathcal{A} = \mathcal{C} \times M_N$ this is the group $\mathcal{U}(N)$ of smooth functions on M^4 with values in the unitary group $U(N)$ and I choose it to be the group of local gauge transformations.

A gauge transformation defines a mapping of $\Omega^1(\mathcal{A})$ onto itself of the form

$$\omega' = g^{-1}\omega g + g^{-1}dg\,. \tag{6.61}$$

We also define

$$\theta' = g^{-1}\theta g + g^{-1}d_V g\,, \qquad A' = g^{-1}Ag + g^{-1}d_H g \tag{6.62}$$

and so ϕ transforms under the adjoint action of $\mathcal{U}(N)$ as

$$\phi' = g^{-1}\phi\; g\,. \tag{6.63}$$

Therefore the θ form remains invariant under the action of these local gauge transformations and the transformed potential ω' is again of the form of (6.60).

The fact that θ is invariant under a gauge transformation means that it cannot be made zero by a choice of gauge. I have then a potential with vanishing curvature but which is not gauge equivalent to zero. If M_N were an algebra of functions over a compact manifold, the existence of a such one-form would be due to the nontrivial topology of the manifold.

I define the curvature two-form Ω and the field strength F as usual

$$\Omega = d\omega + \omega^2, \qquad F = d_H A + A^2\,. \tag{6.64}$$

Having defined the covariant exterior derivative as

$$D\phi = d\phi + \omega\phi + \phi\omega \tag{6.65}$$

and decomposed it into horizontal and vertical parts, the substitution of (6.60) to the first of (6.64) gives

$$\Omega = F + D_H\phi + (D_V\phi - \phi^2)\,. \tag{6.66}$$

In terms of components, with $\phi = \phi_a\theta^a$, $A = A_a\theta^a$ and with definitions

$$\Omega = \frac{1}{2}\Omega_{MN}\theta^M \wedge \theta^N\,, \qquad F = \frac{1}{2}F_{\mu\nu}\theta^\mu\theta^\nu\,, \tag{6.67}$$

I find

$$\begin{aligned} &\Omega_{\mu\nu} = F_{\mu\nu}\,, \qquad \Omega_{\mu a} = D_\mu\phi_a \\ &\Omega_{ab} = [\phi_a, \phi_b] - mC^c{}_{ab}\phi^c\,. \end{aligned} \tag{6.68}$$

I describe the Kaluza-Klein construction in three steps. Firstly one has to identify the internal structure. Using the one-forms θ^M one can consider the algebra $\mathcal{A}$ as the algebra of functions over a manifold of dimension $4+(N^2-1)$: a product of an ordinary manifold of 4 dimensions and an algebraic structure of dimension N^2-1. In general, the invariance group of the complete structure is $SO(3+N^2-1,1)$, but if I restrict to the local rotations which donnot mix the ordinary θ^μ's with the algebraic θ^a's, this group reduces to $SO(3,1)\times SO(N^2-1)$. The group $U(N)$ acts on the algebraic structure through the adjoint representation

$$U(N) \to SO(N^2-1)\,.$$

Only the group $SU(N)/\mathbb{Z}_2$ acts non trivially and I have an embedding

$$SU(N)/\mathbb{Z}_2 \hookrightarrow SO(N^2-1)\,.$$

Therefore, I am forced to suppose that

$$\omega^0{}_a = 0, \qquad A^0{}_\mu = 0\,, \tag{6.69}$$

and that $g \in \mathcal{SU}(N)$, the local gauge transformations.

With the condition (6.69) the connection can be written explicitly as

$$\omega = (A^a{}_\mu\theta^\mu - m\,\theta^a + \phi^a{}_b\theta^b)\lambda_a \tag{6.70}$$

and the derivations of the algebra $\mathcal{A}$ will be

$$\tilde{e}_\mu = e_\mu + \kappa A^a{}_\mu e_a\,, \qquad \tilde{e}_a = e_a + \kappa\omega^b{}_a e_b = \kappa\phi^b{}_a e_b\,. \tag{6.71}$$

Dual to them one defines the one-forms

$$\tilde{\theta}^\mu = \theta^\mu\,, \qquad \tilde{\theta}^a = m\chi^a{}_b(\theta^b - \kappa A^b{}_a\theta^a)\,, \tag{6.72}$$

where the inverse $\chi^a{}_b$ to the matrix $\phi^a{}_b$ have been used.

I firstly describe the special case of connections for which the internal curvature vanishes, i.e. $\Omega_{ab} = 0$. This is the case which is mostly resembles ordinary Kaluza-Klein theory. From (6.68) either the $\phi^a{}_b$ vanishes or $\phi^a{}_b$ belongs to the gauge orbit of $m\delta^a{}_b$. These two cases belong to the two stables vacua of the theory. The set gives rise to a singular set of one-forms $\tilde{\theta}^a$. The second value, which corresponds to the physical vacuum, yields a frame θ^M which is formally similar to the usual moving frame constructed on a principal $SU(N)$ bundle.

I denote the linear connection in $\Omega^1(\mathcal{C})$ as $\omega^\mu{}_\nu$, an $\mathfrak{so}(3,1)$-valued one-form. This has to satisfy the structure equations

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0\,, \qquad d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \Omega^\mu{}_\nu\,. \tag{6.73}$$

Generally I have to construct an $\mathfrak{so}(3+(N^2-1),1)$-valued one-form $\tilde{\omega}^M{}_N$ on $\Omega^1(\mathcal{A})$ satisfying the first structure equation

$$d\tilde{\theta}^M + \omega^M{}_N \wedge \tilde{\theta}^N = 0\,. \tag{6.74}$$

Under the condition of vanishing internal curvature, $\Omega_{ab} = 0$, the solution to these equation is given by

$$\tilde{\omega}^\mu{}_\nu = \omega^\mu{}_\nu + \frac{1}{2}\kappa F_a{}^\mu{}_\nu \tilde{\theta}^a\,, \tag{6.75a}$$

$$\tilde{\omega}^\mu{}_a = \frac{1}{2}\kappa F_a{}^\mu{}_\nu \theta^\nu\,, \tag{6.75b}$$

$$\tilde{\omega}^a{}_b = -\frac{1}{2} m C^a{}_{bc}\tilde{\theta}^c + \kappa C^a{}_{cb} A^c{}_\mu \theta^\mu\,, \tag{6.75c}$$

which except of an additional term in the $\omega^a{}_b$ equation, this connection is formally the same with the one constructed on an $SU(N)$ bundle. The extra term is what remains of the covariant derivative of the Higgs-boson fields.

I consider now a general $SU(N)$ connection with a general Higgs-boson field ($\Omega_{ab} \neq 0$ case). Then the matrix $m\chi^a{}_b$ in (6.72) can be considered as a transformation of the frame $\tilde{\theta}^a$ away from its physical vacuum value. Let me expand the curvature components of the extended space over the algebraic basis of the extra dimensional structure, i.e $\Omega_{MN} = \Omega^a{}_{MN}\lambda_a$ and define for brevity $\Omega'^a{}_{MN} = \chi^a{}_b \Omega^b{}_{MN}$. Then the solutions to (6.74) is given by

$$\tilde{\omega}^\mu{}_\nu = \omega^\mu{}_\nu + \frac{1}{2}\Omega'_a{}^\mu{}_\nu \tilde{\theta}^a\,, \tag{6.76a}$$

$$\tilde{\omega}^\mu{}_a = \frac{1}{2}\Omega'_b{}^\mu{}_a \theta^b + \frac{1}{2}\Omega'_a{}^\mu{}_M \tilde{\theta}^M\,, \tag{6.76b}$$

$$\tilde{\omega}^a{}_b = -\frac{1}{2} m C^a{}_{bc}\tilde{\theta}^c + \frac{1}{2}\Omega'_c{}^a{}_b \tilde{\theta}^c + \frac{1}{2}(\Omega'^a{}_{Mb} - \Omega'_{bM}{}^a)\tilde{\theta}^M\,. \tag{6.76c}$$

As in normal Kaluza-Klein theory the connection contains structure constants and terms which vanish with the curvature.

To complete however the Kaluza-Klein construction, the second structure equation has to be fullfilled by the connection, i.e.

$$d\tilde{\omega}^M{}_N + \tilde{\omega}^M{}_K \wedge \tilde{\omega}^K{}_N = \tilde{\Omega}^M{}_N\,. \tag{6.77}$$

Moreover the equations of motion following from a suitable action have to be also considered. Invariance under local $SO(3,1) \times SU(N)$ transformations permits an infinite sum of terms involving arbitrary powers of $\Omega_{\mu\nu}$, $\Omega_{\mu a}$, Ω_{ab} as well as the Higgs-boson field ϕ_a. It has been shown in [106] that in the usual Kaluza-Klein with an internal space that consistent a classical theory with reasonable stable vacua, only a finite part of the above mention expansion is required. Here I restrict myself to the case of Einstein-Hilbert term leaving aside the discussion of higher order terms in the modified version of Einstein gravity I describe. In that case, one finds for the Lagrangian

$$\begin{aligned}\tilde{R} = R &+ \frac{1}{4}\Omega'_{aMN}\Omega'^{aMN} - \frac{1}{4}m^2 C_{abc}C^{abc} + \frac{1}{2}\Omega'^a{}_{\mu b}\Omega'^{b\mu}{}_a + \Omega'^a{}_{\mu a}\Omega'^{b\mu}{}_b \\ &+ \frac{1}{2}(\Omega'^a{}_{bc} + mC^a{}_{bc})\Omega'^{cb}{}_a\,. \end{aligned}\tag{6.78}$$

All of the terms on the right-hand side of the equation are gauge invariant.

The second term in the equation above is a modified version of the gauge boson Lagrangian. In the next chapter I shall use it to describe a Yang-Mills-Dirac theory over an extended space with noncommutative characteristics; furthermore I shall discuss the dimensional reduction of the theory using generalisation of the CSDR ideas. The third term is an effective cosmological constant. The last three terms do not appear in usual Kaluza-Klein theories. They modify in an essential way the Higgs Lagrangian.

6.5 The fuzzy sphere example as an extra dimensional space

Here, I describe as a concrete example of a matrix geometry the case of the fuzzy sphere. This geometrical construction was proposed in [107]. Thereafter it was considered in a series of different studies as an extra dimensional space [93, 94, 108–114] or as emergent geometry of a matrix model [94, 113, 115, 116]. It is interesting to note that the fuzzy sphere has been also described as the classical stable vacuum of D-branes configurations (Myers effect) [117]. For reviews consult [92, 98, 118].

For the definition of the fuzzy sphere and the gauge theory over it I follow ref. [92] (see also ref. [107]). A fuzzy 'manifold' both in general and in the particular case of the fuzzy sphere is a discrete matrix approximation to the corresponding continuous manifold. These fuzzy spaces are described by associative noncommutative algebras which are constructed by singling-out a finite subspace of the space of functions defined over their corresponding commutative manifolds. This subspace is need to be invariant under multiplication and therefore the discrete matrix approximation must be appropriately

chosen. An essential feature of this approximation is that the discretise space preserves its continuum symmetries [119], in contrast with the lattice gauge theories case. In chapter 7, I will show that this characteristic is of crucial importance for the formulation of the Fuzzy-CSDR scheme.

Turning now to the description of the fuzzy sphere construction, let me concider the $\mathbb{R}^3$ and the commutative coordinates $\tilde{x}^a$, $a = 1, 2, 3$. Let $g_{ab} = \delta_{ab}$ the standard Euclidean metric and the two-sphere manifold, S^2, defined by the constraint

$$g_{ab}\tilde{x}^a\tilde{x}^b = r^2 \,, \tag{6.79}$$

with r being the radius of the sphere. Consider $\mathcal{P}$ the algebra of polynomials in $\tilde{x}^a$ and let $\mathcal{I}$ be the ideal generated by the (6.79). That is the $\mathcal{I}$ consists of elements of $\mathcal{P}$ with a $(g_{ab}\tilde{x}^a\tilde{x}^b - r^2)$ factor. Then the quotient algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is dense in the $\mathcal{C}(S^2)$. Any element of $\mathcal{A}$ can be represented as a finite multipole expansion of the form

$$\tilde{f}(\tilde{x}^a) = f_0 + f_a\tilde{x}^a + \frac{1}{2}f_{ab}\tilde{x}^a\tilde{x}^b + \dots \,, \tag{6.80}$$

where the $f_{a_1,\dots,a_l}$ are completely symmetric and traceless.

A sequence of noncommutative approximation to the $\mathcal{C}(S^2)$ can now be constructed by means of truncating the multipole expansion (6.80) up to a certain order. Indeed, if I truncate all functions to the constant term I reduce the algebra $\mathcal{C}(S^2)$ to the algebra of $\mathcal{A}_1 = \mathbb{C}$ of complex numbers and the geometry of S^2 is reduced to that of a point. Keeping the expansions up to the linear term in the $\tilde{x}^a$ forms a four-dimensional vector space †. However the subspace of monomial functions over $\mathcal{C}(S^2)$ is not invariant under ordinary multiplication. This can be resolved by an appropriate redefinition of the product. Indeed, if I require the radical of $\mathcal{A}_2$ to be equal zero then there are two possible ways-out. First, I can define the product so that $\mathcal{A}_2$ becomes a product of four copies of $\mathbb{C}$. This algebra is commutative and the sphere looks like as a set of four points, being actually a lattice approximation of the original manifold. Alternatively, I can define the product so that $\mathcal{A}_2$ becomes the M_2 algebra of complex 2×2 matrices. That is I replace

$$\tilde{x}^a \mapsto x^a = \bar{k} r^{-1}(\sigma^a/2) \,.$$

The σ^a are the Pauli matrices and the parameter $\bar{k}$ must be related to r by the equation $\bar{k}^2 = \frac{1}{3/4}r^2$ in order the defining constraint of the sphere $g_{ab}x^ax^b = r^2$ to be fullfilled. As I explain in more detail below, the algebra M_2 describes the sphere poorly and it is *fuzzy*. In fact it is a quantum mechanical system with the commutative coordinates replaced by the two-dimensional matrix operators forming an $\mathfrak{su}(2)$ Lie algebra. As such the corresponding geometrical construction is characterised by two eigenvalues distinguishing only the north and south poles of the sphere. However the three operators defining the new geometrical construction cannot be simultaneously measured. Therefore, it is impossible to define the notion of a point over this fuzzy sphere.

† f_a contributes three independent components plus one more from the constant term.

Suppose next that I keep the term quartic in $\tilde{x}^a$. That is I consider the set $\mathcal{A}_3$ of expansions of the form (6.80) with only the coefficients f_0, f_a, f_{ab} nonvanishing. This is a nine-dimensional vector space because of the constraint (6.79). This subspace of functions closes under multiplication by various ways of product redefinition. Among them, I can choose the product in a way that $\mathcal{A}_3$ becomes equal to the algebra M_3 of complex 3×3 matrices and make the replacement

$$\tilde{x}^a \mapsto x^a = \lambda\!\!\!^{-} r^{-1} J^a \,, \tag{6.81}$$

with the J^a being the three-dimensional irrep. of $\mathfrak{su}(2)$ Lie algebra. The sphere is now less fuzzy and the equator as well as the north and south pole can be distinguished.

In the general case one can suppress the N-th order in the $\tilde{x}^a$. The resulting vector space $\mathcal{A}_N$ is of dimensions N^2. It closes under multiplication by a new product in $\tilde{x}^a$ which make it into the algebra of M_N of complex $N \times N$ matrices. Indeed, the number of components of the completely symmetric tensor $f_{a_1 \ldots a_l}$ is given by $\mathrm{N}_l = \binom{N+l-1}{l}$. Because of the constraint (6.79) N_{l-2} of these will not contribute to the expansion (6.80). Therefore there will be $\mathrm{N}_l - \mathrm{N}_{l-2} = 2l+1$ independent monomials of order l and $\sum_{l=1}^{N-1}(2l+1) = N^2$ components in all. Moreover, the $\mathcal{A}_N$ vector space closes if I make the replacement

$$\tilde{x}^a \mapsto x^a = i\lambda\!\!\!^{-} r^{-1} X^a \tag{6.82}$$

where the X^a are considered to be antihermitean and fulfilling the commutation relations

$$[X^a, X^b] = \varepsilon^{ab}{}_c X^c \,, \tag{6.83}$$

that is the N-dimensional irrep. of $\mathfrak{su}(2)$ Lie algebra. The noncommutative coordinates of the geometrical construction will now close under

$$[x^a, x^b] = i\lambda\!\!\!^{-} C^{ab}{}_c x^c \,, \qquad C^{ab}{}_c = r^{-1}\varepsilon^{ab}{}_c \,. \tag{6.84}$$

This product redefinition turns the $\mathcal{A}_N$ vector space to the M_N algebra of complex $N \times N$ matrices. The defining constraint of the sphere is satisfied for the x^a, i.e $g_{ab}x^a x^b = r^2$, if

$$\lambda\!\!\!^{-} = \frac{r^2}{\sqrt{C_2(N)}} \,, \tag{6.85}$$

where $C_2(N) = \frac{1}{4}(N^2-1)$ is the eigenvalue of the second Casimir operator for the case of N-dimensional $\mathfrak{su}(2)$ algebra irrep. Therefore, the (6.82) map takes the form

$$\tilde{x}^a \mapsto x^a = \frac{ir}{\sqrt{C_2(N)}}\, X^a \,. \tag{6.86}$$

Note that for large N the $\lambda\!\!\!^{-} \approx \frac{2r^2}{N}$; for $N \to \infty$ it tends to zero. In this limit, $\lambda\!\!\!^{-} \to 0$, the x^a commute and all of the points of the sphere can be distinguished. In fact, the $\lambda\!\!\!^{-}$ constant is related with the area of elementary cells. Indeed, the state of a particle

on the fuzzy sphere is described as in quantum mechanics by a state vector ψ. An observable associated to the particle is an hermitean element of M_N and the value of the observable f is given by the real number $\psi^* f \psi$. Similarly by what corresponds to the position of the particle is given by the 3 numbers $\psi^* x^a \psi$. As acting on the eigenstate of x^3 with the largest eigenvalue the commutation relations (6.84) become, for large N,

$$[x^1, x^2] = i\bar{k} \,.$$

Therefore, there are elementary cells of area $2\pi\bar{k}$; the $\bar{k}$ itself has dimensions of $(length)^2$ whereas the fuzzy sphere can be covered by $\frac{4\pi r^2}{2\pi\bar{k}} = N$ of them. Since I am considering a noncommutative model of Kaluza-Klein theory I am tempted to identify $\bar{k}$ with the inverse square of the Planck mass, $\bar{k} = \mu_P^{-2}$ and consider the S_N^2 as an extra dimensional space which is fundamentally noncommutative in this length-scale.

The fuzzy sphere construction is more conveniently described in the basis provided by the constant identity matrix $\mathbb{1}$ and the noncommutative spherical harmonics. Suppressing the N-th order in x^a of the multipole expansion (6.80) above, (*fuzzyness level* $N-1$) , these noncommutative spherical harmonics are defined as

$$Y^{lm} = r^{-l} \sum_a c^{lm}_{a_1 \ldots a_l} X^{a_1} \cdots X^{a_l}, \qquad 0 \le l \le N-1$$
$$m = -l, -l+1, \ldots, l-1, l \tag{6.87}$$

with $c^{lm}_{a_1 \cdots a_l}$ the traceless and symmetric tensor of the ordinary spherical harmonics. We choose to normalise these Y^{lm} as

$$\mathrm{Tr}_N \left((Y^{lm})^\dagger Y^{l'm'} \right) = \delta^{ll'} \delta^{mm'} \,. \tag{6.88}$$

A generic function on the fuzzy sphere is expanded under this basis as

$$F = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} Y^{lm} \,, \tag{6.89}$$

or in a more compact form

$$\begin{aligned} S_N^2 \cong (\mathbf{N}) \otimes (\mathbf{N}) &= (\mathbf{1}) \oplus (\mathbf{3}) \oplus \ldots \oplus (\mathbf{N-1}) \\ &= \{Y^{0,0}\} \oplus \ldots \oplus \{Y^{(N-1),m}\} \,, \end{aligned} \tag{6.90}$$

i.e. corresponds to an ordinary function on the commutative sphere with a cut-off on the angular momentum. Obviously this space of truncated functions is closed under the noncommutative $N \times N$ matrix product. Moreover, in the $N \to \infty$ limit one recovers the usual commutative sphere.

A diffeomorphism of S^2 defines and it is defined by an automorphism of the algebra of the smooth functions on S^2, $\mathcal{C}(S^2)$. Then the noncommutative analogue of a diffeomorphism of S^2 is therefore an automorphism of M_N. Since M_N is a simple algebra all of its automorphisms are of the form $f \mapsto f' = g^{-1} f g$ where g is a fixed arbitrary element of

M_N which has an inverse. Since I have considered complex-valued functions on S^2 the algebra $\mathcal{C}(S^2)$ has a $*$-operation, $\tilde{f} \mapsto \tilde{f}^*$ obtained by taking the complex conjugate of $\tilde{f}$. The diffeomorphism of $\mathcal{C}(S^2)$ should respect this $*$-operation: $\tilde{f}^{*\prime} = \tilde{f}^{\prime *}$. It is expected that its corresponding automorphism of M_N will respect this operation too. Therefore for the diffeomorphism is required that $(g^{-1}fg)^\dagger = g^{-1}f^*g$. As a result $g^\dagger = g^{-1}$ and the analogue of diffeomorphisms of S^2 in the noncommutative case are described by

$$x'^a = g^{-1}x^a g, \qquad g \in SU(N)\,. \tag{6.91}$$

A different choice of x^a not connecting the change of coordinates with the above relation would be equivalent to a different choice of a differential or topological structure.

A smooth global vector field on S^2 defines and it is defined by a derivation of the algebra $\mathcal{C}(S^2)$; the noncommutative analogue of a global vector field on S^2 is a derivation of the algebra M_N, namely a linear map X of M_N onto itself satisfying the Leibniz rule (6.2) and the reality condition $(X(f))^* = X(f^*)$. Therefore, as the M_N is a simple algebra the derivations are of the form $X = \mathrm{adj}(h)$ where h is a fixed but arbitrary antihermitean element of M_N. The change of generators (6.91) takes

$$X^a \mapsto X'^a = \mathrm{adj}(g^{-1}hg) \tag{6.92}$$

and so all automorphisms of M_N are analogue of diffeomorphisms of $\mathcal{C}(S^2)$. If g is near identity I can write

$$x'^a \simeq x^a + e^a h, \qquad g \simeq 1 + \frac{h}{i\hbar}\,. \tag{6.93}$$

An important special case is given by $h = h_a x^a$. Then I have

$$x'^a \simeq x^a + C^a{}_{bc} h^b x^c \tag{6.94}$$

and therefore in the limit corresponds to a $\mathbb{R}^3$ rotation around the axis h_a. The formula (6.94) yields the adjoint action of the Lie algebra of $SO(3)$ on M_N, which contains exactly once the irreducible representation of dimension $2j+1$ with $0 \leq j \leq N-1$. The fuzzy sphere construction respects the symmetries of its continuum counterpart.

On the two-sphere there is a natural action of $SU(2)$ which defines three smooth vector fields $\tilde{e}_a$, $a = 1, 2, 3$ which are Killing fields with respect to the induced metric. Correspondingly I single out three derivations of e_a of M_N for every N. The two-sphere is not a parallelizable manifold and the module of derivations $\mathrm{Der}(S^2)$ is not a free module on the three generators $\tilde{e}_a$, namely not all of them are linear independent. They satisfy the relation $\tilde{x}^a \tilde{e}_a = 0$. On the other hand, each of the truncations of the multipole expansion (6.80) makes the geometry of S^2 to look like the geometry of $SU(2)$. In the $N \to \infty$ limit the two pictures coincide as I have already described. The $SU(2)$ covariant differential calculus of the fuzzy sphere is three dimensional. The three derivations e_α along X_α of a function f are given[‡] by

$$e_a(f) = [\mathrm{adj}(X_a)](f) = [X_a, f] = \frac{1}{i\kappa}[x_a, f] \tag{6.95}$$

[‡]The considered metric here is the standard Euclidean one, $g_{ab} = \delta_{ab}$ and I can freely lower the ‘coordinate’ indices.

with $\kappa = \hbar r^{-1}$ and therefore expressed in $(length)^{-1}$ units as in (6.33) of section 6.3. Accordingly the action of the Lie derivatives on functions is given by

$$\mathcal{L}_a f = [X_a, f] = \frac{1}{i\kappa}[x_a, f] \ , \tag{6.96}$$

they satisfy the Leibniz rule and the $SU(2)$ Lie algebra[§] relation

$$[\mathcal{L}_a, \mathcal{L}_b] = \varepsilon^c{}_{ab}\mathcal{L}_c \, . \tag{6.97}$$

In the $N \to \infty$ limit the derivations e_α become

$$\tilde{e}_a = \varepsilon^c{}_{ab} x^b \partial_c \tag{6.98}$$

and only in this commutative limit the tangent space becomes two dimensional. The exterior derivative is given by

$$df = e_a(f)\theta^a = [X_a, f]\theta^a = \frac{1}{i\kappa}[x_a, f]\,\theta^a \tag{6.99}$$

with θ^a the one-forms dual to the vector fields e_a, $< e_a, \theta^b >= \delta_a{}^b$. The space of one-forms is generated by the θ^a's in the sense that for any one-form $\omega = \sum_i f_i(dh_i)\, t_i$ I can always write $\omega = \omega_a \theta^a$ with given functions ω_a depending on the functions f_i, h_i and t_i. From $0 = \mathcal{L}_a(< e_b, \theta^c >) =< \mathcal{L}_a(e_b), \theta^c > + < e_b, \mathcal{L}_a(\theta^c) >$ and $\mathcal{L}_a(e_b) = C_{ab}{}^c e_c$ [cf. (6.97)] I obtain the action of the Lie derivatives on one-forms,

$$\mathcal{L}_a(\theta^b) = \varepsilon_a{}^b{}_c \theta^c \, . \tag{6.100}$$

It is then easy to check that the Lie derivative commutes with the exterior differential d, i.e. $SU(2)$ invariance of the exterior differential. On a general one-form $\omega = \omega_a \theta^a$ I have

$$\begin{aligned} \mathcal{L}_b\omega = \mathcal{L}_b(\omega_a\theta^a) &= (\mathcal{L}_b\omega_a)\theta^a - \omega_a \varepsilon^a{}_{bc}\theta^c \\ &= [X_b, \omega_a]\,\theta^a - \omega_a\varepsilon^a{}_{bc}\theta^c \end{aligned} \tag{6.101}$$

and therefore

$$(\mathcal{L}_b\omega)_a = [X_b, \omega_a] - \omega_c \varepsilon^c{}_{ba} \, . \tag{6.102}$$

Similarly, from $\mathcal{L}_b(v) = \mathcal{L}_b(v^a e_a) = [X_b, v^a]e_a + v^a \mathcal{L}_b(e_\alpha)$ I have

$$(\mathcal{L}_b v)^\alpha = [X_b, v^a] - v^c \varepsilon_{cb}{}^a \, . \tag{6.103}$$

There are different approaches to the study of spinor fields on the fuzzy sphere [120, 121]. Here I follow ref. [92] (section 8.2)[¶]. In the case of the product of Minkowski

[§] This is essentially the Lie algebra under which the angular momentum operators $\mathrm{J}_a = iX_a$ close.

[¶] For a discussion of chiral fermions and index theorems on matrix approximations of manifolds see ref. [122].

space and the fuzzy sphere, $M^4 \times S_N^2$, I have seen that the geometry resembles in some aspects ordinary commutative geometry in seven dimensions. As $N \to \infty$ it returns to the ordinary six-dimensional geometry. Let g_{AB} be the Minkowski metric in seven dimensions and Γ^A the associated Dirac matrices which can be in the form

$$\Gamma^A = (\Gamma^\mu, \Gamma^\alpha) = (1 \otimes \gamma^\mu, \sigma^\alpha \otimes \gamma_5)\,. \tag{6.104}$$

The space of spinors must be a left module with respect to the Clifford algebra. It is therefore a space of functions with values in a vector space $\mathcal{H}'$ of the form

$$\mathcal{H}' = \mathcal{H} \otimes C^2 \otimes C^4,$$

where $\mathcal{H}$ is an M_{N+1} module. The geometry resembles but is not really seven-dimensional, e.g. chirality can be defined and the fuzzy sphere admits chiral spinors. Therefore the space H' can be decomposed into two subspaces $\mathcal{H}'_\pm = \frac{1\pm\Gamma}{2}\mathcal{H}'$, where Γ is the chirality operator of the fuzzy sphere [92, 98]. The same holds for other fuzzy cosets such as $(SU(3)/U(1) \times U(1))_F$ [123].

In order to define the action of the Lie derivative $\mathcal{L}_a$ on a spinor field Ψ, I write

$$\Psi = \zeta_\alpha \psi_\alpha\,, \tag{6.105}$$

where ψ_α are the components of Ψ in the ζ_α basis. Under a spinor rotation $\psi_\alpha \to S_{\alpha\beta}\psi_\beta$ the bilinear $\bar{\psi}\Gamma^a\psi$ transforms as a vector $v^a \to \Lambda_{ab}v^b$. The Lie derivative on the basis ζ_α is given by

$$\mathcal{L}_a \zeta_\alpha = \zeta_\beta \tau^\alpha_{\beta\alpha}\,, \tag{6.106}$$

where

$$\tau^a = \frac{1}{2} C_{abc}\Gamma^{bc}\,, \qquad \Gamma^{bc} = -\frac{1}{4}(\Gamma^b\Gamma^c - \Gamma^c\Gamma^b)\,. \tag{6.107}$$

Using that Γ^{bc} are a rep. of the orthogonal algebra and then using the Jacobi identities for C_{abc} one has $[\tau^a, \tau^b] = C_{abc}\tau^c$ from which it follows that the Lie derivative on spinors gives a representation of the Lie algebra,

$$[\mathcal{L}_a, \mathcal{L}_b]\zeta_\alpha = C_{abc}\,\mathcal{L}_c\,\zeta_\alpha\,. \tag{6.108}$$

On a generic spinor Ψ, applying the Leibniz rule I have

$$\mathcal{L}_a\Psi = \zeta_a[X_a, \psi_\alpha] + \zeta_\beta \tau^\alpha_{\beta\gamma}\psi_\gamma \tag{6.109}$$

and of course $[\mathcal{L}_a, \mathcal{L}_b]\Psi = C_{abc}\,\mathcal{L}_c\,\Psi$; I also write

$$\delta_a\psi_\alpha = (\mathcal{L}_a\Psi)_\alpha = [X_a, \psi_\alpha] + \tau^a_{\alpha\gamma}\psi_\gamma\,. \tag{6.110}$$

The action of the Lie derivative $\mathcal{L}_a$ on the adjoint spinor is obtained considering the adjoint of the above expression, since $(X_a)^\dagger = -X_a$, $(\tau^a)^\dagger = -\tau^a$, $[\tau^a, \Gamma_0] = 0$ one has

$$\delta_a\bar{\psi}_\alpha = [X_a, \bar{\psi}_\alpha] - \bar{\psi}_\gamma \tau^a_{\gamma\alpha}\,. \tag{6.111}$$

One can then check that the variations (6.110) and (6.111) are consistent with $\psi^\dagger\Gamma^0\psi$ being a scalar. Finally I have compatibility among the Lie derivatives (6.110), (6.111) and (6.102):

$$\delta_a(\bar\psi\Gamma^\mu\psi) = [X_a, \bar\psi\Gamma^\mu\psi]\,, \tag{6.112}$$

$$\begin{aligned}\delta_a(\bar\psi\Gamma^d\psi) &= (\delta_a\bar\psi)\Gamma^d\psi + \bar\psi\Gamma^d\delta_a\psi = [X_a, \bar\psi\Gamma^d\psi] + \bar\psi[\Gamma^d, \tau^a]\psi \\ &= [X_a, \bar\psi\Gamma^d\psi] - C_{adc}\bar\psi\Gamma^c\psi\,.\end{aligned} \tag{6.113}$$

This immediately generalises to higher tensors $\bar\psi\,\Gamma_{d_1}\ldots\Gamma_{d_i}\psi$. These relations and the one derived earlier for the forms and the vector fields are fundamental for formulating the CSDR principle on fuzzy cosets. In the next chapter details on these ideas are discussed.

We finally note that the differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S^2_N$, is easily obtained from that on M^4 and on S^2_N. For example a one-form A defined on $M^4 \times S^2_N$ is written as

$$A = A_\mu dx^\mu + A_a\theta^a \tag{6.114}$$

with $A_\mu = A_\mu(x^\mu, X_a)$ and $A_a = A_a(x^\mu, X_a)$.

Matrix approximations of other coset spaces

The S^2 manifold is not the only one that can be approximated by a matrix geometry. This is possible for some other coset spaces too leading to their 'fuzzy-fied' analogues. Such cases has been studied extensively in the recent literature. To be more specific the sphere S^2 is the complex projective space CP^1 . The generalisation of the fuzzy sphere construction to CP^2 and its *spin*^{c} structure was given in ref. [124], whereas the generalisation to $CP^{M-1} = SU(M)/U(M-1)$ and to Grassmannian cosets was given in ref. [119].

While a set of coordinates on the sphere is given by the $\mathbb{R}^3$ coordinates $\tilde{x}^a$ modulo the relation $\sum_a \tilde{x}^a\tilde{x}^a = r^2$, a set of coordinates on CP^{M-1} is given by $\tilde{x}^a$, $a = 1,\ldots M^2-1$ modulo the relations

$$\delta_{ab}\,\tilde{x}^a\tilde{x}^b = \frac{2(M-1)}{M}r^2\,, \qquad d^c{}_{ab}\,\tilde{x}^a\tilde{x}^b = \frac{2(M-2)}{M}r\tilde{x}^c\,, \tag{6.115}$$

where $d^c{}_{ab}$ are the components of the symmetric invariant tensor of $SU(M)$. Then CP^{M-1} is approximated, at fuzziness level N, by $n \times n$ dimensional matrices x_a, $a = 1,\ldots,M^2-1$. These are proportional to the generators X_a of $SU(M)$ considered in the $n = \frac{(M-1+N)!}{(M-1)!N!}$ dimensional irrep., obtained from the N-fold symmetric tensor product of the fundamental M-dimensional representation of $SU(M)$. As before I set $x_a = \frac{1}{ir}X_a$ so that

$$\sum_{a=1}^{3} x_a x_a = -\frac{C_2(n)}{r^2}\,, \qquad [X_a, X_b] = C^c{}_{ab}X_c \tag{6.116}$$

where $C_2(n)$ is the quadratic Casimir of the given n-dimensional irrep., and $rC^c{}_{ab}$ are now the $SU(M)$ structure constants. More generally [123] one can consider fuzzy coset spaces $(S/R)_F$ described by non-commuting coordinates X_a that are proportional to the generators of a given n-dimensional irrep. of the compact Lie group S and thus in particular satisfy the conditions (6.116) where now $rC^c{}_{ab}$ are the S structure constants (the extra constraints associated with the given n-dimensional irrep. determine the subgroup R of S in S/R). The differential calculus on these fuzzy spaces can be constructed as in the case for the fuzzy sphere. For example there are $\dim(S)$ Lie derivatives, they are given by eq. (6.96) and satisfy the relation (6.97). On these fuzzy spaces, the space of spinors is considered to be a left module with respect to the Clifford algebra given by (6.104), where now the σ^a's are replaced by the γ^a's, the gamma matrices on $R^{\dim S}$; in particular all the formulae concerning Lie derivatives on spinors remain unchanged.

Noncommutative gauge fields and transformations

Gauge fields arise in non-commutative geometry and in particular on fuzzy spaces very naturally; they are linked to the notion of covariant coordinate [125]. Consider a field $\phi(X^a)$ on a fuzzy space described by the non-commuting coordinates X^a. An infinitesimal gauge transformation $\delta\phi$ of the field ϕ with gauge transformation parameter $\lambda(X^a)$ is defined by

$$\delta\phi(X) = \lambda(X)\phi(X)\,. \tag{6.117}$$

This is an infinitesimal abelian $U(1)$ gauge transformation if $\lambda(X)$ is just an antihermitian function of the coordinates X^a, it is an infinitesimal nonabelian $U(P)$ gauge transformation if $\lambda(X)$ is valued in $\mathfrak{u}(P)$, the Lie algebra of hermitian $P \times P$ matrices; in the following I will always assume $\mathfrak{u}(P)$ elements to commute with the coordinates X^a. The coordinates X are invariant under a gauge transformation

$$\delta X_a = 0\,; \tag{6.118}$$

multiplication of a field on the left by a coordinate is then not a covariant operation in the non-commutative case. That is

$$\delta(X_a\phi) = X_a\lambda(X)\phi\,, \tag{6.119}$$

and in general the right hand side is not equal to $\lambda(X)X_a\phi$. Following the ideas of ordinary gauge theory one then introduces covariant coordinates φ_a such that

$$\delta(\varphi_a\phi) = \lambda\varphi_a\phi\,, \tag{6.120}$$

this happens if

$$\delta(\varphi_a) = [\lambda, \varphi_a]\,. \tag{6.121}$$

Setting

$$\varphi_a \equiv X_a + A_a \tag{6.122}$$

A_a can be interpreted as the gauge potential of the non-commutative theory; then φ_a is the non-commutative analogue of a covariant derivative. The transformation properties of A_a support the interpretation of A_a as gauge field; they arise from requirement (6.121),

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a] \,. \tag{6.123}$$

Correspondingly one can define a tensor F_{ab}, the analogue of the field strength, as

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C^c{}_{ab} A_c \tag{6.124}$$
$$= [\varphi_a, \varphi_b] - C^c{}_{ab} \varphi_c \,. \tag{6.125}$$

This tensor transforms covariantly

$$\delta F_{ab} = [\lambda, F_{ab}] \,. \tag{6.126}$$

Similarly, for a spinor ψ in the adjoint representation, the infinitesimal gauge transformation is given by

$$\delta\psi = [\lambda, \psi] \,, \tag{6.127}$$

while for a spinor in the fundamental the infinitesimal gauge transformation is given by

$$\delta\psi = \lambda\psi \,. \tag{6.128}$$

Chapter 7

Dimensional Reduction over Fuzzy Coset Spaces

I use the ideas presented in the previous chapter, concerning a specific noncommutative modification of the Kaluza-Klein theory, to describe an interesting generalisation of the Coset Space Dimensional Reduction (CSDR) scheme; this was introduced in [93] and further explored in [111, 112]. One start with a Yang-Mills-Dirac theory defined in a higher dimensional space, $M^D = M^4 \times B$, with the internal space being a coset space which is approximated by a finite matrix algebra, M_N, i.e. a fuzzy coset space, $(S/R)_F$. Having assumed fuzzy coset spaces as the hidden extra dimensions, the theory turns out to be power counting renormalizable; the fuzzy spaces are approximated by matrices of finite dimensions and only a finite number of counterterms are required to make the Lagrangian renormalizable.

In the spirit of noncommutative geometry other particle models with noncommutative gauge theory were explored [126–130] . After the work of Seiberg and Witten [131], where a map (SW map) between noncommutative and commutative gauge theories has been described, there has been a lot of activity also in the construction of noncommutative phenomenological Lagrangians, for example various noncommutative standard model like Lagrangians have been proposed [132–134]*. More recently noncommutative modifications of the SM model based on the spectral triple formalism have been proposed [140–143] and their connection with string theory has been examined [144]. These noncommutative models represent interesting generalisations of the SM and hint at possible new physics. However they do not address the usual problem of the SM, the presence of a plethora of free parameters mostly related to the ad hoc introduction of the Higgs and Yukawa sectors in the theory. In the dimensional reduction scheme which I describe here these two sectors emerge automatically in four dimensions; Higgs particles are the extra dimensional components of the gauge fields defined over the full higher

*These SM actions are mainly considered as effective actions because they are not renormalizable. The effective action interpretation is consistent with the SM in [133, 134] being anomaly free [135]. Noncommutative phenomenology has been discussed in [136–139].

dimensional theory. The Yukawa terms are obtained by the fermions - gauge fields coupling terms. Application of CSDR over fuzzy cosets is tempting for further investigation but the construction of a realistic particle physics model is yet to be explored.

7.1 CSDR over fuzzy coset spaces

The basic idea of the CSDR scheme was described in chapter 2. The solution of the constraints, imposed on the four-dimensional surviving fields was also given.

Here, I consider a higher dimensional gauge theory defined on a compactified space $M^4 \times (S/R)_F$, where $(S/R)_F$ is the approximation of S/R by finite $N \times N$ matrices and have noncommutative characteristics. I denote the local coordinates of the considered extra dimensional space by $x^M = (x^\mu, y^a)$ to resemble the coordinate parametrisation of the compactification over ordinary cosets; here the y^a are proportional with some X_a, $N \times N$ antihermitean matrices. In section 6.5, I recalled the case of fuzzy sphere as an example of a such noncommutative 'manifold'. In a more general set-up one consider a noncommutative gauge theory with gauge group $G = U(P)$ over $M^4 \times (S/R)_F$. The implementation of the CSDR scheme in the fuzzy case (Fuzzy-CSDR)- following [93] - can be described in three steps:

1. State the CSDR principle on fuzzy cosets and reduce it to a set of constraints - the CSDR constraints (7.7), (7.9), (7.12), (7.15), (7.16) - that the gauge and matter fields must satisfy.
2. Reinterpret a Yang-Mills-Dirac action on $M^4 \times (S/R)_F$ with $G = U(P)$ gauge group as actions on M^4 with $U(NP)$ gauge group. This is possible by expanding the fields on $M^4 \times (S/R)_F$ in Kaluza-Klein modes on $(S/R)_F$. The algebra of functions on $(S/R)_F$ is finite dimensional and one obtain a finite tower of modes; the $(S/R)_F$ is described by $N \times N$ matrices and a basis for this mode expansion is given by the generators of Lie algebra $\mathfrak{u}(N)$. It has been proven that the different modes can be conveniently grouped together so that an initial $\mathfrak{g}$-valued field on $M^4 \times (S/R)_F$ (with $G = U(P)$) is reinterpreted as a $\mathfrak{u}(NP)$ valued field on M^4.
3. Solve the CSDR constraints and obtain the gauge group and the particle content of the reduced four-dimensional actions. I present the example of dimensional reduction over the two-dimensional fuzzy sphere and describe its generalisation over fuzzy cosets of more than 2 dimensions.

7.2 The CSDR principle

Since the Lie algebra of S acts on the fuzzy space $(S/R)_F$, one can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be

invariant under the infinitesimal S action up to an infinitesimal gauge transformation

$$\mathcal{L}_b \phi = \delta^{W_b} \phi = W_b \phi \qquad \mathcal{L}_b A = \delta^{W_b} A = -DW_b\,, \tag{7.1}$$

where A is the one-form gauge potential $A = A_\mu(x^\mu, y^a)dx^\mu + A_a(x^\mu, y^a)\theta^a$, and W_b depends only on the coset coordinates $y^a \sim X_a$ and (like A_μ, A_a) is antihermitean. I thus write $W_b = W_b^{(\alpha)} \mathcal{T}^\alpha$, $\alpha = 1, 2 \ldots P^2$, where $\mathcal{T}^\alpha$ are hermitean generators of $U(P)$ and $(W_b^{(\alpha)})^\dagger = -W_b^{(\alpha)}$, here † is hermitean conjugation on the X_a's. The principle gives for the space-time part A_μ [c.f. (6.96)]

$$\mathcal{L}_b A_\mu = [X_b, A_\mu] = -[A_\mu, W_b]\,, \tag{7.2}$$

while for the internal part A_a [c.f. (6.102)]

$$[X_b, A_a] - A_c C^c{}_{ba} = -[A_a, W_b] - \mathcal{L}_a W_b\,. \tag{7.3}$$

Taking in account the cyclicity condition of the Lie derivatives $[\mathcal{L}_a, \mathcal{L}_b] = C^c{}_{ab}\mathcal{L}_c$, and that from the first of eqs. (7.1) I have $\mathcal{L}_a \mathcal{L}_b \phi = (\mathcal{L}_a W_b)\phi + W_b W_a \phi$ which lead to the consistency condition

$$[X_a, W_b] - [X_b, W_a] - [W_a, W_b] = C^c{}_{ab} W_c\,. \tag{7.4}$$

Under the gauge transformation $\phi \to \phi^{(g)} = g\,\phi$ with $g \in G = U(P)$, I have $\mathcal{L}_a \phi^{(g)} = W_a^{(g)} \phi^{(g)}$ and also $\mathcal{L}_a \phi^{(g)} = (\mathcal{L}_a g)\,\phi + g\,(\mathcal{L}_a \phi)$, and therefore

$$W_\alpha \to W_a^{(g)} = g\, W_a\, g^{-1} + [X_a, g]\, g^{-1}\,. \tag{7.5}$$

Now in order to solve the constraints (7.2), (7.3), (7.4) I cannot follow the strategy adopted in the commutative case where the constraints were studied just at one point of the coset (say $y^a = 0$). This is due to the intrinsic nonlocality of the constraints. On the other hand the specific properties of the fuzzy case (e.g. the fact that partial derivatives are realised via commutators, the concept of covariant derivative) allow to simplify and eventually solve the constraints. Defining

$$\omega_a \equiv X_a - W_a\,, \tag{7.6}$$

one obtain the following form of the consistency condition (7.4)

$$[\omega_a, \omega_b] = C^c{}_{ab}\, \omega_c\,, \tag{7.7}$$

where ω_a transforms as

$$\omega_\alpha \to \omega_a^{(g)} = g\, \omega_a\, g^{-1}\,. \tag{7.8}$$

Now eq. (7.2) reads

$$[\omega_b, A_\mu] = 0\,. \tag{7.9}$$

Furthermore by considering the covariant coordinate,

$$\varphi_d \equiv X_d + A_d \tag{7.10}$$

one has

$$\varphi \rightarrow \varphi^{(g)} = g\,\varphi\,g^{-1} \tag{7.11}$$

and eq. (7.3) simplifies to

$$[\omega_b, \varphi_a] = \varphi_c C^c{}_{ba}\,. \tag{7.12}$$

Therefore eqs. (7.7), (7.9) and (7.12) are the constraints to be solved. Note that eqs. (7.11) and (7.12) have the symmetry

$$\varphi_a \rightarrow \varphi_a + \omega_a\,, \tag{7.13}$$

suggesting that ω_a is a ground state and φ_a the fluctuations around it. Indeed, the semi-positive definite potential (7.20), vanishes for the value $\varphi_a^{(vac)} = \omega_a$.

One proceeds in a similar way for the spinor fields. The CSDR principle relates the Lie derivative on a spinor ψ, which is considered here to transform in the adjoint representation of G, to a gauge transformation; recalling eqs. (6.107) and (6.110) one has

$$[X_a, \psi] + \frac{1}{2}C_{abc}\Gamma^{bc}\psi = [W_a, \psi]\,, \tag{7.14}$$

where ψ denotes the column vector with entries ψ_α. Setting again $\omega_a = X_a - W_a$ lead to the constraint

$$-\frac{1}{2}C_{abc}\Gamma^{bc}\psi = [\omega_a, \psi]\,. \tag{7.15}$$

Having considered spinors which transform in the fundamental rep. of the gauge group G, one has $[X_a, \psi] + \frac{1}{2}C_{abc}\Gamma^{bc}\psi = W_a\psi,$. Setting again $\omega_a = X_a - W_a$, lead to the constraint

$$-\frac{1}{2}C_{abc}\Gamma^{bc}\psi = \omega_a\psi - \psi X_a\,. \tag{7.16}$$

7.3 Actions and Kaluza-Klein modes

Here, I consider a pure Yang-Mills action on $M^4 \times (S/R)_F$ and recall how it is reinterpreted in four dimensions. The action is

$$\mathcal{A}_{YM} = \frac{1}{4}\int d^4x\, \mathrm{Tr}\, tr_G\, F_{MN}F^{MN}\,, \tag{7.17}$$

where Tr is the usual trace over $N \times N$ matrices and is actually the integral over the fuzzy coset $(S/R)_F$ †, while tr_G is the gauge group G trace. The higher dimensional field strength F_{MN} decomposed in four-dimensional space-time and extra dimensional components reads as follows $(F_{\mu\nu}, F_{\mu b}, F_{ab})$; explicitly the various components of the field strength are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \\ F_{\mu a} &= \partial_\mu A_a - [X_a, A_\mu] + [A_\mu, A_a] = \partial_\mu \varphi_a + [A_\mu, \varphi_a] = D_\mu \varphi_a\,, \\ F_{ab} &= [\varphi_a, \varphi_b] - C^c{}_{ab}\varphi_c\,; \end{aligned} \tag{7.18}$$

they are covariant under local G transformations: $F_{MN} \to g\, F_{MN}\, g^{-1}$, with $g = g(x^\mu, X^a)$.

In terms of the suggested decomposition the action reads

$$\mathcal{A}_{YM} = \int d^4x \,\mathrm{Tr}\, tr_G \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(D_\mu \varphi_a)^2 \right) - V(\varphi)\,, \tag{7.19}$$

where the potential term $V(\varphi)$ is the F_{ab} kinetic term (recall F_{ab} is antihermitean so that $V(\varphi)$ is hermitean and non-negative)

$$\begin{aligned} V(\phi) &= -\frac{1}{4}\mathrm{Tr}\, tr_G \sum_{ab} F_{ab}F_{ab} \\ &= -\frac{1}{4}\mathrm{Tr}\, tr_G \sum_{ab} \left([\varphi_a, \varphi_b] - C^c{}_{ab}\varphi_c\right)\left([\varphi_a, \varphi_b] - C^c{}_{ab}\varphi_c\right)\,. \end{aligned} \tag{7.20}$$

The action (7.19) is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^\mu, X^\alpha)$ can indeed be interpreted just as an M^4 gauge transformation. I write

$$\lambda(x^\mu, X^a) = \lambda^\alpha(x^\mu, X^a)\mathcal{T}^\alpha = \lambda^{\alpha,h}(x^\mu)T^h\mathcal{T}^\alpha\,, \tag{7.21}$$

where $\mathcal{T}^\alpha$ are hermitean generators of $U(P)$, $\lambda^\alpha(x^\mu, X^a)$ are $N \times N$ antihermitean matrices; these can be expanded in finite symmetric multipole expansion over X's [c.f. section 6.5] and are expressible as $\lambda(x^\mu)^{\alpha,h}T^h$, where T^h are antihermitean generators of $U(N)$. The fields $\lambda(x^\mu)^{\alpha,h}$, with $h = 1, \dots N^2$, are the Kaluza-Klein modes (KK-modes) of $\lambda(x^\mu, X^a)^\alpha$. Then one considers on equal footing the indices h and α and interprets the fields on the r.h.s. of (7.21) as one field valued in the tensor product Lie algebra $\mathfrak{u}(N) \otimes \mathfrak{u}(P)$. This Lie algebra is indeed $\mathfrak{u}(NP)$ ‡. Similarly, I can rewrite the gauge field A_ν as

$$A_\nu(x^\mu, X^a) = A_\nu^\alpha(x^\mu, X^a)\mathcal{T}^\alpha = A_\nu^{\alpha,h}(x^\mu)T^h\mathcal{T}^\alpha\,, \tag{7.22}$$

† Tr is a good integral because it has the cyclic property $\mathrm{Tr}(f_1 \dots f_{p-1}f_p) = \mathrm{Tr}(f_p f_1 \dots f_{p-1})$. It is also invariant under the action of the group S, that I recall to be infinitesimally given by $\mathcal{L}_a f = [X_a, f]$.

‡Proof: The $(NP)^2$ generators $T^h\mathcal{T}^\alpha$ are $NP \times NP$ antihermitean matrices. Then one just have to show that they are linearly independent. This is easy since it is equivalent to prove the linear independence of the $(NP)^2$ matrices $e_{ij}\varepsilon_{\rho\sigma}$ where $i = 1, \dots n$, $\rho = 1, \dots P$ and e_{ij} is the $N \times N$ matrix having 1 in the position (i, j) and zero elsewhere, and similarly for the $P \times P$ matrix $\varepsilon_{\rho\sigma}$.

and interpret it as a $\mathfrak{u}(NP)$ valued gauge field on M^4, and similarly for φ_a. Finally $\mathrm{Tr}\, tr_G$ is the trace over $U(NP)$ matrices in the fundamental representation.

The above analysis applies also to more general actions, and to the field ω_a and therefore to the CSDR constraints (7.7), (7.9), (7.12), (7.15), (7.16) that can now be reinterpreted as constraints on M^4 instead of on $M^4 \times (S/R)_F$. The action (7.19) and the minima of the potential (7.20), in the case $P = 1$, have been studied, without CSDR constraints, in refs. [145–149].

7.4 CSDR constraints for the fuzzy sphere

Here, I present the solution of the aforementioned CSDR constraints for the case of two-dimensional fuzzy sphere and extend the results to more general fuzzy cosets. I consider $(S/R)_F = S^2_N$, i.e. the two-dimensional fuzzy sphere approximated by $N \times N$ matrices (fuzziness level $N-1$). I first examine the simpler case where the gauge group G is just $U(1)$ and I make some comments on the $G = U(P)$ generalisation afterwards.

The $G = U(1)$ case

In this case the $\omega_a(X_b)$ that appear in the consistency condition (7.7) are $N \times N$ anti-hermitean matrices, i.e. I can interpret them as elements of the $\mathfrak{u}(N)$ Lie algebra. On the other hand eqs. (7.7) are the commutation relations of the $\mathfrak{su}(2)$ Lie algebra. Indeed, according to section 6.5, if r is the radius of the fuzzy sphere, the structure constants entering in the CSDR constraints of section 7.2 are given by $C^a{}_{bc} = r^{-1}\varepsilon^a{}_{bc}$. Then $r\omega_a$ are the $\mathfrak{su}(2)$ Lie generators in a reducible or in the N-dim irreducible rep. and define an $\mathfrak{su}(2)$ image into the $\mathfrak{u}(N)$ Lie algebra.

The four-dimensional gauge symmetry is determined by solving the constraint (7.9). We consider the expansion of the $A_\mu(x, X)$ into the Kaluza-Klein modes of the S^2_N. Recalling from the previous section that in the simplest case of $G = U(1)$ gauge group, A_μ is reinterpreted as a four-dimensional $\mathfrak{u}(N)$-valued field, an embedding of $\mathfrak{su}(2) \hookrightarrow \mathfrak{u}(N)$ is required. One possible embedding is the following. Let T^h with $h = 1, \ldots, N^2$ be the generators of $\mathfrak{u}(N)$ in the fundamental representation and with normalisation $Tr(T^h T^k) = -\frac{1}{2}\delta^{hk}$. These appear in the expansion $A_\mu(x, X) = A^h_\mu(x) T^h$. We can always use the convention $h = (a, u)$ with $a = 1, 2, 3$ and $u = 4, 5, \ldots, N^2$ where the T^a satisfy the $\mathfrak{su}(2)$ Lie algebra

$$[T^a, T^b] = rC^{ab}{}_c T^c \,. \tag{7.23}$$

Then I define an embedding by identifying

$$r\omega_a = T_a \,, \tag{7.24}$$

i.e. a regular $\mathfrak{su}(2)$ subalgebra of $\mathfrak{u}(N)$. Constraint (7.9), $[\omega_b, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centraliser of the image of $SU(2)$ in $U(N)$,

i.e.

$$K = C_{U(N)}(SU(2)) = SU(N-2) \times U^{I}(1) \times U^{II}(1)\,, \tag{7.25}$$

where $U(N) \simeq SU(N) \times U^{II}(1)$. The functions $A_\mu(x)$ are arbitrary functions of x and take values in Lie $\mathfrak{K}$ subalgebra of $\mathfrak{u}(N)$.

Concerning constraint (7.12), $[\omega_b, \varphi_a] = C^c{}_{ba}\varphi_c$, it is satisfied by choosing

$$\varphi_a = \varphi(x) r \omega_a \tag{7.26}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ that is a singlet under the four-dimensional gauge group K.

The physical spinor fields (transforming in the adjoint rep.) are obtained by solving the constraint (7.15), $-\frac{1}{2}C_{abc}\Gamma^{bc}\psi = [\omega_a, \psi]$. In the l.h.s. of this formula one can say that she (he) has an embedding of $\mathfrak{su}(2)$ in the spin representation of $\mathfrak{so}(3)$. This embedding is given by the matrices $\tau^a = \frac{1}{2}C^a{}_{bc}\Gamma^{bc}$; since $\mathfrak{su}(2) \sim \mathfrak{so}(3)$ this embedding is rather trivial and indeed $\tau^a = \frac{-i}{2r}\sigma^a$. Thus the constraint (7.15) states that the spinor $\psi = \psi^h T^h = \binom{\psi_1}{\psi_2}$ where $T^h \in \mathfrak{u}(N)$ and $\psi_{1(2)} = \psi^h_{1(2)}T^h$ are four-dimensional spinors, relate (intertwine) the fundamental rep. of $SU(2)$ to the representations of $SU(2)$ induced by the embedding (7.24) of $SU(2)$ into $U(N)$, i.e. of $SU(2)$ into $SU(N)$. In formulae

$$\begin{aligned} SU(N) &\supset SU(2) \times SU(N-2) \times U(1) \\ \mathbf{N^2-1} &= (\mathbf{1},\mathbf{1})_{(0)} \oplus (\mathbf{3},\mathbf{1})_{(0)} \oplus (\mathbf{1},(\mathbf{N-2})^{\mathbf{2}})_{(0)} \\ &\quad \oplus (\mathbf{2},(\mathbf{N-2}))_{(-N)} \oplus (\mathbf{2},\overline{(\mathbf{N-2})})_{(N)}\,. \end{aligned}$$

Therefore, the fermions that satisfy constraint (7.15) transform as $(\mathbf{N-2})_{(-N,0)}$ and $\overline{(\mathbf{N-2})}_{(N,0)}$ under $K = SU(N-2) \times U^I(1) \times U^{II}(1)$. In the case of the fuzzy sphere the embedding $\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(3)$ is somehow trivial. If I had chosen instead the fuzzy $(SU(3)/U(1) \times U(1))_F$, then $\mathfrak{su}(3)$ should be embedded in $\mathfrak{so}(8)$.

In order to write the action for fermions I have to consider the Dirac operator $\mathcal{D}$ on $M^4 \times S^2_N$. This operator can be constructed following the derivation presented in ref. [120] for the Dirac operator on the fuzzy sphere, see also ref. [98]. For fermions in the adjoint I obtain

$$\mathcal{D}\psi = i\Gamma^\mu(\partial_\mu + A_\mu)\psi + i\sigma^a[X_a + A_a, \psi] - \frac{1}{r}\psi\,, \tag{7.27}$$

where Γ^μ is defined in (6.104), and with slight abuse of notation I have written σ^a instead of $\sigma^a \otimes 1$. Using eq. (7.10) the fermion action,

$$\mathcal{A}_F = \int d^4x \,\mathrm{Tr}\,\bar\psi \mathcal{D}\psi \tag{7.28}$$

becomes

$$\mathcal{A}_F = \int d^4x \;\; \mathrm{Tr}\,\bar\psi \left(i\Gamma^\mu(\partial_\mu + A_\mu) - \frac{1}{r} \right)\psi \; + i\,\mathrm{Tr}\,\bar\psi\sigma^a[\phi_a, \psi]\,, \tag{7.29}$$

where I recognise the fermion masses $1/r$ and the Yukawa interactions.

Using eqs. (7.26), (7.15) the Yang-Mills action (7.19) plus the fermion action reads

$$\begin{aligned}\mathcal{A}_{YM}+\mathcal{A}_F &= \int d^4x\ \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})-\frac{3}{4}D_\mu\varphi D^\mu\varphi-\frac{3}{8}(\varphi^2-r^{-1}\varphi)^2\\ &+\int d^4x\ \mathrm{Tr}\,\bar\psi\left(i\Gamma^\mu(\partial_\mu+A_\mu)-\frac{1}{r}\right)\psi-\frac{3}{2}\,\mathrm{Tr}\,\bar\psi\varphi\psi\,. \end{aligned}\tag{7.30}$$

The choice (7.24) defines one of the possible embeddings of $\mathfrak{su}(2)\hookrightarrow\mathfrak{u}(N)$ [$\mathfrak{su}(2)$ is embedded in $\mathfrak{u}(N)$ as a regular subalgebra], while on the other extreme one can embed $\mathfrak{su}(2)$ in $\mathfrak{u}(N)$ using the irreducible N dimensional rep. of $SU(2)$

$$T_a = r\omega_a = X_a^{(N)}\,. \tag{7.31}$$

Constraint (7.9) in this case implies that the four-dimensional gauge group is $U(1)$ so that $A_\mu(x)$ is $U(1)$ valued. Constraint (7.12) leads again to the scalar singlet $\varphi(x)$. A different $\mathfrak{su}(2)\hookrightarrow\mathfrak{u}(N)$ embedding can be described by the choice

$$T_a = r\omega_a = X_a^{(N')}\otimes\mathbb{1}_n \qquad N=N'n\,. \tag{7.32}$$

and the surviving four-dimensional gauge group is found to be $U(n)\times U(1)$ as it can be proven by the explicit calculation of the massless modes of the A_μ Kaluza-Klein expansion. In the next section, I give details of the calculation. Note that the extra $U(1)$ comes from the $U(N)\simeq SU(N)\times U(1)$ as the example I studied before. Obviously, the $\mathfrak{su}(2)\hookrightarrow\mathfrak{u}(N)$ embedding of eq. (7.31) is a special case of the one described by eq. (7.32).

Summarising, the surviving four dimensional spinors are given by the (7.15) constraint. Constraint (7.12) gives the surviving four-dimensional scalars $\varphi_a=\varphi(x)\,r\omega_a$, which is in fact the N-dim irreducible or a reducible rep. of $\mathfrak{su}(2)$. Note that the semi-positive definite potential is always minimised by this value, a result which is independent of the chosen $SU(2)\hookrightarrow U(N)$ embedding.

One can also consider the case of a Yang-Mills-Dirac actions with fermions transforming in the fundamental of the gauge group G. Details of the relevant calculation can be found in [93].

The $G=U(P)$ case

In this case $A_\mu(x,X)=A_\mu^{\alpha,h}(x)T^h\mathcal{T}^\alpha$ is an $NP\times NP$ antihermitean matrix and in order to solve the constraint (7.7) one has to embed $\mathfrak{su}(2)\hookrightarrow\mathfrak{u}(NP)$. All the results of the $G=U(1)$ case holds also here, I just have to replace N with NP. This is true for the fermion sector too, provided that in the higher dimensional theory the fermions are considered in the adjoint of $U(P)$ (then, in the action (7.28) I need to replace Tr with $\mathrm{Tr}\,tr_{U(P)}$ i.e. $tr_{U(NP)}$). The case of Yang-Mills-Dirac action with fermion transforming in the fundamental of the gauge group $G=U(P)$ has been also examined in [93].

7.5 Kaluza-Klein modes on S_N^2 and symmetry breaking

To see the above arguments in more detail let me determine the spectrum and the representation content of the gauge field A_μ in the simplest case for a gauge group $G = U(1)$. The obtained conclusions are easily generalised also for the case of the gauge group $G = U(P)$. Assuming the embedding $\mathfrak{su}(2) \hookrightarrow \mathfrak{u}(N)$, (7.31), I show that the $U(N)$ gauge symmetry, which the full action functional has after its KK expansion, breaks spontaneously to $K = U(1)$ in four dimensions [cf (7.9)]. I expand the $M^4 \times S_N^2$ fields over the KK modes of the extra dimensional fuzzy sphere. It turns out that the CSDR surviving fields are no other than the massless KK modes.

Since the X_a are considered to be the generators of the fuzzy sphere S_N^2, I can decompose the full extra-dimensional $\mathfrak{u}(N)$-valued gauge fields A_μ into spherical harmonics $Y^{lm}(X)$ on the fuzzy sphere S_N^2 with coordinates X_a:

$$A_\mu = A_\mu(x, X) = \sum_{\substack{0\le l\le N\\ |m|\le l}} A_{\mu,lm}(x) \otimes Y^{lm(N)}(X)\,; \tag{7.33}$$

$Y^{lm(N)}$ are by definition irreps under the $SU(2)$ rotations on S_N^2, and form a basis of hermitean $N \times N$ matrices. The $A_{\mu,lm}(x)$ are taken here to be $\mathfrak{u}(1)$-valued gauge and vector fields on M^4. Using this expansion, I can interpret $A_\mu(x, X)$ as $\mathfrak{u}(N)$-valued functions on $M^4 \times S_N^2$, expanded into the Kaluza-Klein modes (i.e. harmonics) of S_N^2.

The scalar fields φ_a,

$$\varphi_a(x, X) = \frac{1}{r}\, X_a^{(N)} + A_a(x, X)\,, \tag{7.34}$$

with potential (7.20) are considered as 'covariant coordinates' on S_N^2 (section 7.2) and take the value $\omega_a = (1/r)X_a^{(N)}$ in vacuum. On the other hand, the fluctuations A_a of these covariant coordinates should be interpreted as $\mathfrak{u}(N)$ gauge fields on the fuzzy sphere, (see appendix B.1). Therefore they can be expanded similarly as

$$A_a = A_a(x, X) = \sum_{\substack{0\le l\le N\\ |m|\le l}} A_{a,lm}(x) \otimes Y^{lm(N)}(X) \tag{7.35}$$

and interpreted as functions (or one-form) on $M^4 \times S_N^2$ taking values in $\mathfrak{u}(N)$. One can then interpret $A_M(x, X) = (A_\mu(x, X), A_a(x, X))$ as $\mathfrak{u}(N)$-valued gauge or vector fields on $M^4 \times S_N^2$.

Larger gauge groups are possible to be retrieved in four dimensions by considering different $\mathfrak{su}(2)$ into $\mathfrak{u}(N)$ embeddings. Indeed, having assumed the $\mathfrak{su}(2) \hookrightarrow \mathfrak{u}(N)$ embedding, (7.32), the 'covariant coordinate' of the two-dimensional fuzzy sphere would be

$$\varphi_a(x, X) = \frac{1}{r}\,(X_a^{(N')} \otimes \mathbb{1}_n) + A_a(x, X)\,. \tag{7.36}$$

The consistency condition (7.7) and the CSDR constraints (7.9), (7.12) and (7.15) remain unmodified. The four and extra dimensional components of the $A_M(x, X)$ gauge field are expanded as

$$A_\mu = A_\mu(x, X) = \sum_{\substack{0 \le l \le N \\ |m| \le l}} A^{(n)}_{\mu,lm}(x) \otimes Y^{lm(N')}(X) \tag{7.37}$$

$$A_a = A_a(x, X) = \sum_{\substack{0 \le l \le N \\ |m| \le l}} A^{(n)}_{a,lm}(x) \otimes Y^{lm(N')}(X)\,, \tag{7.38}$$

where $Y^{lm(N')}$ are N'-dim irreps, under the $SU(2)$ rotations on S^2_N. The $A^{(n)}_{\mu,lm}(x)$ and $A^{(n)}_{a,lm}(x)$ are taken here to be $\mathfrak{u}(n)$-valued gauge and vector fields on M^4. As before then, the $A_M(x, X) = (A_\mu(x, X), A_a(x, X))$ is interpreted as $\mathfrak{u}(N)$-valued ($N = N'n$) gauge or vector fields on $M^4 \times S^2_N$.

Given this expansion into KK modes, I will show that only $A_{\mu,00}(x)$ (i.e. the dimensionally reduced gauge field) becomes a massless $\mathfrak{u}(1)$ or $\mathfrak{u}(n)$-valued gauge field in four dimensions, depending on the 'covariant coordinate' configuration I consider [(7.34) or (7.36)]. All other modes $A_{\mu,lm}(X)$ with $l \ge 1$ constitute a tower of Kaluza-Klein modes with large mass gap, and decouple for low energies.

The scalar fields $A_a(x, X)$ will be analysed in a similar way below, and provide no additional massless degrees of freedom in four dimensions. The surviving fields in four dimensions, found by the previously considered dimensional reduction of the ten-dimensional Yang-Mills-Dirac theory, are exactly the massless modes of their KK expansion. Remarkably, the model I describe is fully renormalizable in spite of its higher-dimensional character, in contrast to the commutative case; see also [112].

Computation of the KK masses

To justify these claims, let me compute the masses of the KK modes (7.33). They are induced by the covariant derivatives $\int \mathrm{Tr}\,(D_\mu \varphi_a)^2$ in (7.19),

$$\int \mathrm{Tr}\,(D_\mu\varphi_a)^\dagger D_\mu\varphi_a = \int \mathrm{Tr}\,(\partial_\mu\varphi_a^\dagger\partial_\mu\varphi_a + 2(\partial_\mu\varphi_a^\dagger)[A_\mu,\varphi_a] + [A_\mu,\varphi_a]^\dagger[A_\mu,\varphi_a])\,. \tag{7.39}$$

The most general scalar field configuration is given by (7.36). As usual, the last term in (7.39) leads to the mass terms for the gauge fields A_μ in the vacuum

$$\varphi_a^{(vac)} = \omega_a = \frac{1}{r}\left(X_a^{(N')} \otimes \mathbb{1}_n\right)\,, \tag{7.40}$$

provided the mixed term which is linear in A_μ vanishes in a suitable gauge. This is usually achieved by going to the unitary gauge. In the present case this is complicated by the fact that I have three scalars in the adjoint, and there is no obvious definition of the unitary gauge; in fact, there are are too many scalar degrees of freedom as to gauge away that term completely. However, I can choose a gauge where all quadratic

contributions of that term vanish, leaving only cubic interaction terms. To see this, I insert (7.36) into the term $(\partial_\mu \varphi_a^\dagger)[A_\mu, \varphi_a]$ in (7.39), which gives

$$\int \mathrm{Tr}\, A_\mu[\varphi_a, \partial_\mu \varphi_a^\dagger] = -\int \mathrm{Tr}\, A_\mu \Big\{[\widehat{X}_a, \partial_\mu A_a(x,X)] + [A_a(x,X), \partial_\mu A_a(x,X)]\Big\}\,,$$

where $\widehat{X}_a = X_a^{(N')} \otimes \mathbb{1}_n$. Now I partially fix the gauge by imposing the 'internal' Lorentz gauge $[\widehat{X}_a, A_a] = 0$ at each point X. This is always possible[§], and the above simplifies as

$$\int \mathrm{Tr}\, A_\mu[\varphi_a, \partial_\mu \varphi_a^\dagger] = \int \mathrm{Tr}\, A_\mu[A_a(x,y), \partial_\mu A_a(x,y)] =: S_{int}\,. \tag{7.41}$$

This contains only cubic interaction terms, which are irrelevant for the computation of the masses. I can therefore proceed by setting the vacuum of the model $\varphi_a^{(vac)}$ [relation (7.40)] and inserting the expansion (7.37) of A_μ into the last term of (7.39). Recall that $\mathrm{J}_a = iX_a$ are the angular momentum generators, satisfying $[\mathrm{J}_a, \mathrm{J}_b] = i\varepsilon^c{}_{ab}\mathrm{J}_c$. Then since

$$i[\widehat{X}_a, A_\mu] = \mathrm{J}_a A_\mu = \sum_{l,m} \left(A_{\mu,lm}^{(n)}(x) \otimes \mathrm{J}_a Y^{lm(N')} \right)$$

is simply the action of $SU(2)$ on the fuzzy sphere, it follows that $\mathrm{Tr}\,[X_a, A_\mu][X_a, A_\mu]$ is the quadratic Casimir on the modes of A_μ which are orthogonal, and I obtain

$$\int \mathrm{Tr}\,(D_\mu \varphi_a)^\dagger D_\mu \varphi_a = \int \mathrm{Tr}\Big(\partial_\mu \varphi_a^\dagger \partial_\mu \varphi_a + \frac{1}{r^2}\sum_{l,m} l(l+1)\, A_{\mu,lm}^{(n)}(x)^\dagger A_{\mu,lm}^{(n)}(x)\Big) + S_{int}\,. \tag{7.42}$$

Therefore the four-dimensional $\mathfrak{u}(n)$ gauge fields $A_{\mu,lm}^{(n)}(x)$ acquire a mass

$$m_l^2 = \frac{1}{r^2}\, l(l+1)\,, \tag{7.43}$$

with r being the radius of the fuzzy sphere and therefore of compactification energy scale, as it is expected for higher KK modes. In particular, only $A_\mu^{(n)}(x) \equiv A_{\mu,00}^{(n)}(x)$ survives as a massless four-dimensional $\mathfrak{u}(n)$ gauge field. The low-energy effective action (LEA) for the gauge sector is then given by

$$S_{LEA}^{(gauge)} = \int d^4x\, \frac{1}{4g^2}\, tr_K\, F_{\mu\nu}^\dagger F_{\mu\nu}\,, \tag{7.44}$$

where $F_{\mu\nu}$ is the field strength of the low-energy $A_\mu(x)$gauge fields. These take values in the Lie algebra generating the $K = SU(n) \times U^I(1) \times U^{II}(1)$ gauge group which describe the symmetry of the theory in four dimensions. The abelian group factors comes from $U(N) \simeq SU(N) \times U^{II}(1)$ and $U(n) \simeq SU(n) \times U^I(1)$. Obviously by setting $n = 1$ in the

[§]Even though this gauge is commonly used in the literature on the fuzzy sphere, a proof of existence has apparently not been given. It can be proved by extremizing the real function $\mathrm{Tr}\,(X_a\varphi_a)$ on a given gauge orbit, which is compact; the e.o.m. then implies $[X_a, \varphi_a] = 0$.

partition of $N = N'n$, I obtain the same result but with the surviving four-dimensional gauge group to be $K = U(1)$.

Scalar sector

I now expand the most general scalar fields φ_a into modes, singling out the coefficient of the 'radial mode' as

$$\varphi_a(x, X) = X_a^{(N')} \otimes \Big(\frac{1}{r} \mathbb{1}_n + \varphi(x) \Big) + \sum_k A_{a,k}(X) \otimes \varphi_k(x) \,. \tag{7.45}$$

Here $A_{a,k}(x)$ stands for a suitable basis labeled by k of fluctuation modes of gauge fields on S_N^2, and $\varphi(x)$ resp. $\varphi_k(x)$ are $\mathfrak{u}(n)$-valued. One expects that all fluctuation modes in the expansion (7.45) have a large mass gap of the order of the KK scale, which is indeed the case as shown in detail in the B.2 appendix. Therefore she (he) can drop all these modes for the low-energy sector. However, the field $\varphi(x)$ plays a somewhat special role. It corresponds to fluctuations of the radius of the internal fuzzy sphere, which is the order parameter responsible for the SSB $SU(N) \to SU(n)$, and assumes the value $\mathbb{1}_n$ in (7.45). $\varphi(x)$ is therefore the Higgs which acquires a positive mass term in the broken phase, which can be obtained by inserting

$$\varphi_a(x, X) = X_a^{(N')} \otimes \Big(\frac{1}{r} \mathbb{1}_n + \varphi(x) \Big)$$

into $V(\varphi)$, (7.20). Then the Higgs potential is found to be

$$V(\varphi) = \frac{1}{2} N C_2(N')(1 - r^{-1})^2 \Big(\varphi^2(x) + 2r^{-1}\varphi(x) + r^{-2} \Big) \,. \tag{7.46}$$

In conclusion the model presented here behaves like a $U(n)$ gauge theory on $M^4 \times S_N^2$, with the expected tower of KK modes on the fuzzy sphere S_N^2 of radius r. The Yang-Mills part of the low-energy effective action is given by the lowest KK mode, which is

$$\begin{aligned} S_{LEA} = \int d^4x \, tr_K \Big[& \frac{1}{4g^2} F^\dagger_{\mu\nu} F_{\mu\nu} + N C_2(N') \, D_\mu \varphi(x) D_\mu \varphi(x) \\ & - \frac{1}{2} N C_2(N')(1 - r^{-1})^2 \Big(\varphi^2(x) + 2r^{-1}\varphi(x) + r^{-2} \Big) \Big] + S_{int} \end{aligned} \tag{7.47}$$

for the $U(n)$ gauge field $A_\mu(y) \equiv A_{\mu,00}(y)$.

7.6 CSDR constraints for fuzzy cosets

Consider a fuzzy coset $(S/R)_F$ (e.g. fuzzy CP^M) described by $n \times n$ matrices, and let the higher dimensional theory have gauge group $U(P)$. Then constraint (7.7) forms the

Lie algebra of the isometry group S of the coset realized in its n-dim irrep. To solve the (7.9) constraint an embedding of S in $U(nP)$ has to be defined. As a result, the four-dimensional gauge group K is the centraliser of the image $S_{U(nP)}$ of S in $U(nP)$, $K = C_{U(nP)}(S_{U(nP)})$.

Concerning fermions in the adjoint, in order to solve constraint (7.15) one considers the embedding

$$S \hookrightarrow SO(\dim(S)) \ ,$$

which is given by $\tau_a = \frac{1}{2}C_{abc}\Gamma^{bc}$ that satisfies $[\tau_a, \tau_b] = C_{abc}\tau^c$. Therefore ψ is an intertwining operator between induced representations of S in $U(nP)$ and in $SO(\dim(S))$. To find the surviving fermions, as in the commutative case [18], one has to decompose the adjoint rep. of $U(nP)$ under $S_{U(nP)} \times K$,

$$\begin{aligned} &U(nP) \supset S_{U(nP)} \times K \\ &\mathrm{adj}[U(nP)] = \sum_i (s_i, k_i) \end{aligned} \tag{7.48}$$

and the spinor rep. σ of $SO(\dim(S))$ under S

$$\begin{aligned} &SO(\dim(S)) \supset S \\ &\sigma = \sum_e \sigma_e \ . \end{aligned} \tag{7.49}$$

Then, for two identical irreps. $s_i = \sigma_e$, there is a k_i multiplet of fermions surviving in four dimensions, i.e. four-dimensional spinors $\psi(x)$ belonging to the k_i representation of K.

7.7 Discussion

I discussed here a generalisation of the CSDR scheme over spaces approximated by the algebra of finite matrices (fuzzy cosets). I followed a rather modest approach keeping the commutative nature of ordinary spacetime and assume only the internal space to be a noncommutative one, i.e. a fuzzy space. The main advantage of this assumption is that even the higher dimensional theory is renormalizable. Due to the finite dimension of the matrices approximating the fuzzy spaces only a finite number of counterterms is required for the renormalizability of the theory. Furthermore the Higgs and Yukawa sectors of a simple particle physics model resulted from the dimensional reduction itself which is an intrinsic characteristic of the CSDR scheme. To be more specific, as in ordinary CSDR case, Higgs particles are the extra dimensional components of the gauge fields of the initially defined ten-dimensional Yang-Mills-Dirac theory. The Yukawa terms are obtained by the coupling terms between fermions and gauge fields.

However, the Fuzzy-CSDR has some different features from the ordinary CSDR and leads to new possibilities to build reasonable low-energy theories which however remain

to be investigated. A major difference between fuzzy and ordinary CSDR is that in the fuzzy case one always embeds S in the gauge group G instead of embedding just R in G. This is due to the fact that the differential calculus used in the Fuzzy-CSDR is based on $\dim(S)$ derivations instead of the restricted $\dim(S) - \dim(R)$ used in the ordinary one. As a result the four-dimensional gauge group $H = C_G(R)$ appearing in the ordinary CSDR after the geometrical breaking and before the spontaneous symmetry breaking due to the four-dimensional Higgs fields does not appear in the Fuzzy-CSDR. In Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR constraints. The four-dimensional potential has the typical 'maxican hat' shape, but it appears already spontaneously broken. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Correspondingly in the Yukawa sector of the theory one has the results of the spontaneous symmetry breaking, i.e. massive fermions and Yukawa interactions among fermions and the physical Higgs field. Having massive fermions in the final theory is a generic feature of CSDR when S is embedded in G [18]. Therefore, if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a $U(1)$ in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions.

Chapter 8

Dynamical generation of Fuzzy Extra Dimensions and Symmetry Breaking

According to the discussion so far, CSDR over fuzzy coset spaces, $(S/R)_F$, leads to four-dimensional theories with phenomenologically interesting characteristics. Theories defined on the assumed compactified space, $M^4 \times (S/R)_F$, were found to be renormalizable. Motivated by the interesting features of this approach, I examine here the inverse problem, i.e. whether obtaining fuzzy extra dimensions as a vacuum solution of a four-dimensional but renormalizable potential is possible. Indeed, starting from the most general renormalizable potential in four dimensions, fuzzy extra dimensions is dynamically generated [94] as the energetically preferred solution . Furthermore the initial gauge symmetry was found to break spontaneously towards phenomenologically interesting patterns.

In the following sections after a small introduction, I consider a renormalizable four-dimensional $SU(\mathcal{N})$ gauge theory with a suitable multiplet of scalar fields which dynamically develop extra dimensions in the form of a two-dimensional fuzzy sphere, S^2_N. The potential of the theory is chosen to be the most general renormalizable in four dimensions and its minimum is calculated for some part of the parameter space (section 8.2). The tower of massive Kaluza-Klein modes is consistent with an interpretation as gauge theory on $M^4 \times S^2_N$, the scalars being interpreted as gauge fields on S^2_N. The gauge group is broken dynamically, and the low-energy content of the model is determined. Depending on the parameters of the model the low-energy gauge group can be $SU(n)$, or broken further to $SU(n_1) \times SU(n_2) \times U(1)$, with mass scale determined by the size of the extra dimension (section 8.3). Finally, I make some remarks on the results of the work and their connection with the Fuzzy-CSDR discussed in the previous chapter.

8.1 Introduction

In the following sections, I consider a renormalizable $SU(\mathcal{N})$ gauge theory on four-dimensional Minkowski space M^4, containing three scalars in the adjoint of $SU(\mathcal{N})$ that transform as vectors under an additional global $SO(3)$ symmetry with the most general renormalizable potential. Then it can be proven that the model dynamically develops fuzzy extra dimensions, more precisely a two-dimensional fuzzy sphere S^2_N. The appropriate interpretation is therefore as gauge theory on $M^4 \times S^2_N$. The low-energy effective action is that of a four-dimensional gauge theory on M^4, whose gauge group and field content is dynamically determined by compactification and dimensional reduction on the internal sphere S^2_N. An interesting and quite rich pattern of spontaneous symmetry breaking (SSB) appears, breaking the original $SU(\mathcal{N})$ gauge symmetry down to much smaller and potentially quite interesting low-energy gauge groups. In particular, I find explicitly the tower of massive Kaluza-Klein states, which justifies the interpretation as a compactified higher-dimensional gauge theory. Nevertheless, the model is renormalizable.

The effective geometry, the symmetry breaking pattern and the low-energy gauge group are determined dynamically in terms of a few free parameters of the potential. Here, I discuss in detail the two simplest possible vacua with gauge groups $SU(n)$ and $SU(n_1) \times SU(n_2) \times U(1)$. I find explicitly the tower of massive Kaluza-Klein modes corresponding to the effective geometry. The mass scale of these massive gauge bosons is determined by the size of the extra dimensions, which in turn depends on some logarithmically running coupling constants. In the case of the $SU(n_1) \times SU(n_2) \times U(1)$ vacuum, I identify in particular massive gauge fields in the bifundamental, similar as in GUT models with an adjoint Higgs. Moreover, I also identify a candidate for a further symmetry breaking mechanism, which may lead to a low-energy content of the theory close to the standard model.

Perhaps the most remarkable aspect of our model is that the geometric interpretation and the corresponding low-energy degrees of freedom depend in a nontrivial way on the parameters of the model, which are running under the RG group. Therefore the massless degrees of freedom and their geometrical interpretation depend on the energy scale. In particular, the low-energy gauge group generically turns out to be $SU(n_1) \times SU(n_2) \times U(1)$ or $SU(n)$, while gauge groups which are products of more than two simple components [apart from $U(1)$] do not seem to occur in this model. Moreover, the values of n_1 and n_2 are determined dynamically, and may well be small such as 3 and 2. A full analysis of the hierarchy of all possible vacua and their symmetry breaking pattern is not trivial however, and remain to be investigated. Here, I restrict myself to establish the basic mechanisms and features of the model, and discuss in the two following sections the two simplest cases (named as 'type 1' and 'type 2' vacuum) in some detail.

The construction under discussion was further developed by the addition of fermions in the action [150]. In particular, in the vacua with low-energy gauge group $SU(n_1) \times SU(n_2) \times U(1)$, the extra-dimensional sphere always carries a magnetic flux with nonzero

monopole number leading to chiral massless fermions. Unfortunately this not possible for a minimal set of fermions and their spectrum have to be doubled.

The idea to use fuzzy spaces for the extra dimensions is certainly not new. The work I describe here was motivated by the Fuzzy-CSDR approach discussed in the previous chapter and combined with lessons from the matrix-model approach to gauge theory on the fuzzy sphere [151, 152]. This leads in particular to a dynamical mechanism of determining the vacuum, SSB patterns and background fluxes. A somewhat similar model has been studied recently in [153, 154], which realises deconstruction and a 'twisted' compactification of an extra fuzzy sphere based on a supersymmetric gauge theory. The model under discussion is different and does not require supersymmetry, leading to a much richer pattern of symmetry breaking and effective geometry. For other relevant work see e.g. [149].

The dynamical formation of fuzzy spaces found here is also related to recent work studying the emergence of stable submanifolds in modified IIB matrix models. In particular, previous studies based on actions for fuzzy gauge theory different from ours generically only gave results corresponding to $U(1)$ or $U(\infty)$ gauge groups, see e.g. [155–157] and references therein. The dynamical generation of a nontrivial index on noncommutative spaces has also been observed in [158, 159] for different models.

The mechanism under discussion may also be very interesting in the context of the recent observation [160] that extra dimensions are very desirable for the application of noncommutative field theory to particle physics. Other related recent work discussing the implications of the higher-dimensional point of view on symmetry breaking and Higgs masses can be found in [61, 161–163]. These issues could now be discussed within a renormalizable framework.

Finally the dynamical or spontaneous generation of extra dimensions occurring in the construction I present here is strongly suggestive of gravity. Indeed the results of [164] allow to understand this mechanism in terms of gravity: the scalar potential defines a matrix-model action which - using a slight generalisation of [164] - can be interpreted as non-abelian Yang-Mills coupled to dynamical Euclidean gravity in the extra dimensions.

8.2 The four-dimensional action

Let a $SU(\mathcal{N})$ gauge theory on four-dimensional Minkowski space M^4 with coordinates x^μ, $\mu = 0, 1, 2, 3$. The action under consideration is

$$\mathcal{S}_{YM} = \int d^4x\, Tr \left(\frac{1}{4g^2} F^\dagger_{\mu\nu} F_{\mu\nu} + (D_\mu \phi_a)^\dagger D_\mu \phi_a \right) - V(\phi) \tag{8.1}$$

where A_μ are $\mathfrak{su}(\mathcal{N})$-valued gauge fields, $D_\mu = \partial_\mu + [A_\mu, .]$, and

$$\phi_a = -\phi_a^\dagger , \qquad a = 1, 2, 3 \tag{8.2}$$

are three antihermitian scalars transforming under the adjoint action of $SU(\mathcal{N})$,

$$\phi_a \to U^\dagger \phi_a U \tag{8.3}$$

where $U = U(x) \in SU(\mathcal{N})$. Furthermore, the ϕ_a transform as vectors of an additional global $SO(3)$ symmetry. The potential $V(\phi)$ is taken to be the most general renormalizable action invariant under the above symmetries, which is

$$\begin{aligned} V(\phi) &= Tr\,(g_1\phi_a\phi_a\phi_b\phi_b + g_2\phi_a\phi_b\phi_a\phi_b - g_3\varepsilon_{abc}\phi_a\phi_b\phi_c + g_4\phi_a\phi_a) \\ &\quad + \frac{g_5}{\mathcal{N}}Tr(\phi_a\phi_a)Tr(\phi_b\phi_b) + \frac{g_6}{\mathcal{N}}Tr(\phi_a\phi_b)Tr(\phi_a\phi_b) + g_7\,. \end{aligned} \tag{8.4}$$

This may not look very transparent at first sight, however it can be written in a very intuitive way. First, I make the scalars dimensionless by rescaling $\phi_a \to \phi_a/R$ where R has dimension of length; I will usually suppress R since it can immediately be reinserted. Then for suitable choice of R

$$R = \frac{2g_2}{g_3}\,, \tag{8.5}$$

the potential can be rewritten as

$$V(\phi) = Tr\left(a^2(\phi_a\phi_a + \tilde{b}\,\mathbb{1})^2 + c + \frac{1}{\tilde{g}^2}F^\dagger_{ab}F_{ab}\right) + \frac{h}{\mathcal{N}}g_{ab}g_{ab} \tag{8.6}$$

for suitable constants $a, b, c, \tilde{g}, h$, where

$$\begin{aligned} &F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c = \varepsilon_{abc}F_c\,, \\ &\tilde{b} = b + \frac{d}{\mathcal{N}}Tr(\phi_a\phi_a)\,, \\ &g_{ab} = Tr(\phi_a\phi_b)\,. \end{aligned} \tag{8.7}$$

I shall omit c from now. The potential is clearly semi-positive definite provided

$$a^2 = g_1 + g_2 > 0\,, \qquad \frac{2}{\tilde{g}^2} = -g_2 > 0\,, \qquad h \geq 0\,, \tag{8.8}$$

which I assume from now on. Here $\tilde{b} = \tilde{b}(x)$ is a scalar, $g_{ab} = g_{ab}(x)$ is a symmetric tensor under the global $SO(3)$, and $F_{ab} = F_{ab}(x)$ is a $\mathfrak{su}(\mathcal{N})$-valued antisymmetric tensor field which will be interpreted as field strength in some dynamically generated extra dimensions below. In this form, $V(\phi)$ looks like the action of Yang-Mills gauge theory on a fuzzy sphere in the matrix formulation [151, 152, 165, 166]. The presence of the first term $a^2(\phi_a\phi_a + \tilde{b})^2$ might seem strange at first, however I should not simply omit it since it would be reintroduced by renormalisation. In fact it is necessary for the interpretation as Yang-Mills action, [151, 166], and I shall show that it is very welcome on physical grounds since it dynamically determines and stabilises a vacuum, which can be interpreted as extra-dimensional fuzzy sphere. In particular, it removes unwanted flat directions.

Let me briefly comment on the RG flow of the various constants. Without attempting any precise computations here, one can see by looking at the potential (8.4) that g_4 will be quadratically divergent at one loop, while g_1 and g_2 are logarithmically divergent. Moreover, the only diagrams contributing to the coefficients g_5, g_6 of the 'nonlocal' terms are nonplanar, and thus logarithmically divergent but suppressed by $\frac{1}{\mathcal{N}}$ compared to the other (planar) diagrams. This justifies the explicit factors $\frac{1}{\mathcal{N}}$ in (8.4) and (8.7). Finally, the only one-loop diagram contributing to g_3 is also logarithmically divergent. In terms of the constants in the potential (8.6), this implies that R, a, $\tilde{g}$, d and h are running logarithmically under the RG flux, while b and therefore $\tilde{b}$ is running quadratically. The gauge coupling g is of course logarithmically divergent and asymptotically free.

A full analysis of the RG flow of these parameters is complicated by the fact that the vacuum and the number of massive resp. massless degrees of freedom depends sensitively on the values of these parameters, as will be discussed below. This indicates that the RG flow of this model will have a rich and nontrivial structure, with different effective description at different energy scales.

8.2.1 The minimum of the potential

Let me try to determine the minimum of the potential (8.6). This turns out to be a rather nontrivial task, and the answer depends crucially on the parameters in the potential.

For suitable values of the parameters in the potential, one can immediately write down the vacuum. Assume for simplicity $h = 0$ in (8.6) . Since $V(\phi) \geq 0$, the global minimum of the potential is certainly achieved if

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c \ = 0, \qquad -\phi_a\phi_a = \tilde{b}\,, \tag{8.9}$$

because then $V(\phi) = 0$. This implies that ϕ_a is a rep. of $SU(2)$, with prescribed Casimir* $\tilde{b}$. These equations may or may not have a solution, depending on the value of $\tilde{b}$. Assume first that $\tilde{b}$ coincides with the quadratic Casimir of finite-dimensional irrep. of $SU(2)$,

$$\tilde{b} = C_2(N) = \frac{1}{4}(N^2 - 1) \tag{8.10}$$

for some $N \in \mathbb{N}$. If furthermore the dimension $\mathcal{N}$ of the matrices ϕ_a can be written as

$$\mathcal{N} = Nn\,, \tag{8.11}$$

then clearly the solution of (8.9) is given by

$$\phi_a = X_a^{(N)} \otimes 1\!\!1_n \tag{8.12}$$

up to a gauge transformation, where $X_a^{(N)}$ denote the generator of the N-dimensional irrep. of $SU(2)$. This can be viewed as a special case of (8.14) below, consisting of n copies of the irrep. $\mathbf{N}$ of $SU(2)$.

*Note that $-\phi \cdot \phi = \phi^\dagger \cdot \phi > 0$ since the fields are antihermitian.

For generic $\tilde{b}$, the equations (8.9) cannot be satisfied for finite-dimensional matrices ϕ_a. The exact vacuum (which certainly exists since the potential is positive definite) can in principle be found by solving the 'vacuum equation' $\frac{\delta V}{\delta \phi_a} = 0$,

$$a^2 \left\{ \phi_a, (\phi \cdot \phi + \tilde{b}) + \frac{d}{\mathcal{N}} Tr(\phi \cdot \phi + \tilde{b}) \right\} + \frac{2h}{\mathcal{N}} g_{ab}\phi_b + \frac{1}{\tilde{g}^2} \left(2[F_{ab}, \phi_b] + F_{bc}\varepsilon_{abc}\right) = 0 \quad (8.13)$$

where $\phi \cdot \phi = \phi_a \phi_a$. We note that all solutions under consideration will imply $g_{ab} = \frac{1}{3}\delta_{ab} Tr(\phi \cdot \phi)$, simplifying this expression.

The general solution of (8.13) is not known. However, it is easy to write down a large class of solutions: any decomposition of $\mathcal{N} = n_1 N_1 + ... + n_k N_k$ into irreps of $SU(2)$ with multiplicities n_i leads to a block-diagonal solution

$$\phi_a = diag\Big(\alpha_1\, X_a^{(N_1)} \otimes \mathbb{1}_{n_1}, \ldots, \alpha_k\, X_a^{(N_k)} \otimes \mathbb{1}_{n_k}\Big) \quad (8.14)$$

of the vacuum equations (8.13), where α_i are suitable constants which will be determined below. There are hence several possibilities for the true vacuum, i.e. the global minimum of the potential. Since the general solution is not known, I proceed by first determining the solution of the form (8.14) with minimal potential, and then discuss a possible solution of a different type ('type 3 vacuum').

Type 1 vacuum. It is clear that the solution with minimal potential should satisfy (8.9) at least approximately. It is therefore plausible that the solution (8.14) with minimal potential contains only representations (reps) whose Casimirs are close to $\tilde{b}$. In particular, let N be the dimension of the irrep. whose Casimir $C_2(N) \approx \tilde{b}$ is closest to $\tilde{b}$. If furthermore the dimensions match as $\mathcal{N} = Nn$, I expect that the vacuum is given by n copies of the irrep. $\mathbf{N}$, which can be written as

$$\phi_a = \alpha\, X_a^{(N)} \otimes \mathbb{1}_n \,. \quad (8.15)$$

This is a slight generalisation of (8.12), with α being determined through the vacuum equations (8.13),

$$a^2(\alpha^2 C_2(N) - \tilde{b})(1 + d) + \frac{h}{3}\,\alpha^2 C_2(N) - \frac{1}{\tilde{g}^2}\,(\alpha - 1)(1 - 2\alpha) = 0 \quad (8.16)$$

A vacuum of the form (8.15) will be denoted as 'type 1 vacuum'. As I will explain in detail, it has a natural interpretation in terms of a dynamically generated extra-dimensional fuzzy sphere S_N^2, by interpreting $X_a^{(N)}$ as generator of a fuzzy sphere (c.f. section 6.5). Furthermore, I will show in section 8.3.1 that this type 1 vacuum (8.15) leads to spontaneous symmetry breaking, with low-energy (unbroken) gauge group $SU(n)$. The low-energy sector of the model can then be understood as compactification and dimensional reduction on this internal fuzzy sphere.

Let me discuss equation (8.16) in more detail. It can of course be solved exactly, but an expansion around $\alpha = 1$ is more illuminating. To simplify the analysis I assume

$d = h = 0$ from now on, and assume furthermore that $a^2 \approx (1/\tilde{g}^2)$ have the same order of magnitude[†]. Defining the *real* number $\tilde{N}$ by $\tilde{b} = \frac{1}{4}(\tilde{N}^2 - 1)$, one finds

$$\alpha = 1 - \frac{m}{N} + \frac{m(m+1)}{N^2} + O(\frac{1}{N^3}) \qquad \text{where } m = N - \tilde{N} \tag{8.17}$$

assuming N to be large and m small. Notice that a does not enter to leading order. This can be understood by noting that the first term in (8.16) is dominating under these assumptions, which determines α to be (8.17) to leading order. The potential $V(\phi)$ is then dominated by the term

$$\frac{1}{\tilde{g}^2} F^\dagger_{ab} F_{ab} = \frac{1}{2\tilde{g}^2} m^2 \, \mathbb{1} \; + O(\frac{1}{N}) , \tag{8.18}$$

while $(\phi_a \phi_a + \tilde{b})^2 = O(\frac{1}{N^2})$. There is a deeper reason for this simple result: If $\tilde{N} \in \mathbb{N}$, then the solution (8.15) can be interpreted as a fuzzy sphere $S^2_{\tilde{N}}$ carrying a magnetic monopole of strength m, as shown explicitly in [151]; see also [167, 168]. Then (8.18) is indeed the action of the monopole field strength.

Type 2 vacuum. It is now easy to see that for suitable parameters, the vacuum will indeed consist of several distinct blocks. This will typically be the case if $\mathcal{N}$ is not divisible by the dimension of the irrep. whose Casimir is closest to $\tilde{b}$.

Consider again a solution (8.14) with n_i blocks of size $N_i = \tilde{N} + m_i$, assuming that $\tilde{N}$ is large and $\frac{m_i}{\tilde{N}} \ll 1$. Generalising (8.18), the action is then given by

$$V(\phi) = Tr\Big(\frac{1}{2\tilde{g}^2} \sum_i n_i \, m_i^2 \, \mathbb{1}_{N_i} \; + O(\frac{1}{N_i})\Big) \approx \frac{1}{2\tilde{g}^2} \frac{\mathcal{N}}{k} \sum_i n_i \, m_i^2 \tag{8.19}$$

where $k = \sum n_i$ is the total number of irreps, and the solution can be interpreted in terms of 'instantons' (nonabelian monopoles) on the internal fuzzy sphere [151]. Hence in order to determine the solution of type (8.14) with minimal action, one simply has to minimise $\sum_i n_i \, m_i^2$, where the $m_i \in \mathbb{Z} - \tilde{N}$ satisfy the constraint $\sum n_i \, m_i = \mathcal{N} - k\tilde{N}$.

It is now easy to see that as long as the approximations used in (8.19) are valid, the vacuum is given by a partition consisting of blocks with no more than two distinct sizes N_1, N_2 which satisfy $N_2 = N_1 + 1$. This follows from the convexity of (8.19): assume that the vacuum is given by a configuration with 3 or more different blocks of size $N_1 < N_2 < \ldots < N_k$. Then the action (8.19) could be lowered by modifying the configuration as follows: reduce n_1 and n_k by one, and add 2 blocks of size $N_1 + 1$ and $N_k - 1$. This preserves the overall dimension, and it is easy to check (using convexity) that the action (8.19) becomes smaller. This argument can be applied as long as there are 3 or more different blocks, or 2 blocks with $|N_2 - N_1| \geq 2$. Therefore if $\mathcal{N}$ is large,

[†]Otherwise the vacuum of the theory cannot be stabilised among other flat directions of the potential and the good characteristics of the matrix model I consider are spoiled.

the solution with minimal potential among all possible partitions (8.14) is given either by a type 1 vacuum, or takes the form

$$\phi_a = \begin{pmatrix} \alpha_1 X_a^{(N_1)} \otimes \mathbb{1}_{n_1} & 0 \\ 0 & \alpha_2 X_a^{(N_2)} \otimes \mathbb{1}_{n_2} \end{pmatrix}, \tag{8.20}$$

where the integers N_1, N_2 satisfy

$$\mathcal{N} = N_1 n_1 + N_2 n_2 , \qquad N_2 = N_1 + 1 . \tag{8.21}$$

A vacuum of the form (8.20) will be denoted as 'type 2 vacuum', and is the generic case. In particular, the integers n_1 and n_2 are determined dynamically. This conclusion might be altered for nonzero d, h or by a violation of the approximations used in (8.19). I shall show in subsection 8.3.2 that this type of vacuum leads to a low-energy (unbroken) gauge group $SU(n_1) \times SU(n_2) \times U(1)$, and the low-energy sector can be interpreted as dimensional reduction of a higher-dimensional gauge theory on an internal fuzzy sphere, with features similar to a GUT model with SSB $SU(n_1+n_2) \to SU(n_1) \times SU(n_2) \times U(1)$ via an adjoint Higgs. Furthermore, since the vacuum (8.20) can be interpreted as a fuzzy sphere with nontrivial magnetic flux [151], one can expect to obtain massless chiral fermions in the low-energy action [150].

In particular, it is interesting to see that gauge groups which are products of more than two simple components [apart from $U(1)$] do not occur in this model. Furthermore one can easily verify that in cases in which both partitions of $\mathcal{N}$ are possible, namely $\mathcal{N} = Nn$ and $\mathcal{N} = n_1 N_1 + n_2 N_2$, the latter is energetically preferable. A numerical study concerning the stability of vacua type 1 and type 2 is presented in appendix B.3.

Type 3 vacuum. Finally, it could be that the vacuum is of a type different from (8.14), e.g. with off-diagonal corrections such as

$$\phi_a = \begin{pmatrix} \alpha_1 X_a^{(N_1)} \otimes \mathbb{1}_{n_1} & \varphi_a \\ -\varphi_a^\dagger & \alpha_2 X_a^{(N_2)} \otimes \mathbb{1}_{n_2} \end{pmatrix} \tag{8.22}$$

for some small φ_a. I shall provide evidence for the existence of such a vacuum below, and argue that it leads to a further SSB. This might play a role similar to low-energy ('electroweak') symmetry breaking, which will be discussed in more detail below. In particular, it is interesting to note that the φ_a will no longer be in the adjoint of the low-energy gauge group. A possible way to obtain a SSB scenario close to the standard model is discussed in subsection 8.3.4.

8.2.2 Emergence of extra dimensions and the fuzzy sphere

Before discussing these vacua and the corresponding symmetry breaking in more detail, I want to explain the geometrical interpretation, assuming first that the vacuum has the

form (8.15). The $X_a^{(N)}$ are then interpreted as coordinate functions (generators) of a fuzzy sphere S_N^2, and the 'scalar' action

$$S_\phi = TrV(\phi) = Tr\Big(a^2(\phi_a\phi_a + \tilde{b})^2 + \frac{1}{\tilde{g}^2}\,F_{ab}^\dagger F_{ab}\Big) \tag{8.23}$$

for $\mathcal{N} \times \mathcal{N}$ matrices ϕ_a is precisely the action for a $U(n)$ Yang-Mills theory on S_N^2 with coupling $\tilde{g}$, as shown in [151] and reviewed in appendix B.1. In fact, the 'unusual' term $(\phi_a\phi_a + \tilde{b})^2$ is essential for this interpretation, since it stabilises the vacuum $\phi_a = X_a^{(N)}$ and gives a large mass to the extra 'radial' scalar field which otherwise arises. The fluctuations of $\phi_a = X_a^{(N)} + A_a$ then provide the components A_a of a higher-dimensional gauge field $A_M = (A_\mu, A_a)$, and the action (8.1) can be interpreted as Yang-Mills theory on the six-dimensional space $M^4 \times S_N^2$, with gauge group depending on the particular vacuum. Note that e.g. for the 'type 1 vacuum', the local gauge transformations $U(\mathcal{N})$ can indeed be interpreted as local $U(n)$ gauge transformations on $M^4 \times S_N^2$.

In other words, the scalar degrees of freedom ϕ_a conspire to form a fuzzy space in extra dimensions. Therefore one interprets the vacuum (8.15) as describing dynamically generated extra dimensions in the form of a fuzzy sphere S_N^2, with an induced Yang-Mills action on S_N^2. This geometrical interpretation will be fully justified in section 8.3 by working out the spectrum of Kaluza-Klein modes. The effective low-energy theory is then given by the zero modes on S_N^2, which is analogous to the models considered in [93]. However, in the present approach one has a clear dynamical selection of the geometry due to the first term in (8.23).

It is interesting to recall here the running of the coupling constants under the RG as discussed above. The logarithmic running of R implies that the scale of the internal spheres is only mildly affected by the RG flow. However, $\tilde{b}$ is running essentially quadratically, hence is generically large. This is quite welcome here: starting with some large $\mathcal{N}$, $\tilde{b} \approx C_2(\tilde{N})$ must indeed be large in order to lead to the geometric interpretation discussed above. Hence the problems of naturalness or fine-tuning appear to be rather mild here.

8.3 Kaluza-Klein modes, dimensional reduction, and symmetry breaking

I now study the model (8.1) in more detail. Let me emphasise again that this is a four-dimensional renormalizable gauge theory, and there is no fuzzy sphere or any other extra-dimensional structure to start with. I have already discussed possible vacua of the potential (8.23), depending on the parameters $a, \tilde{b}, \tilde{g}$ and $\mathcal{N}$. This is a nontrivial problem, the full solution of which is beyond the discussion of this dissertation. I restrict myself here to the simplest types of vacua discussed in subsection 8.2.1, and derive some of the properties of the resulting low-energy models, such as the corresponding low-energy gauge groups and the excitation spectrum. In particular, I exhibit the tower

of Kaluza-Klein modes in the different cases. This turns out to be consistent with an interpretation in terms of compactification on an internal sphere, demonstrating without a doubt the emergence of fuzzy internal dimensions. In particular, the scalar fields ϕ_a become gauge fields on the fuzzy sphere.

8.3.1 Type 1 vacuum and $SU(n)$ gauge group

Let me start with the simplest case, assuming that the vacuum has the form (8.15). I want to determine the spectrum and the rep. content of the gauge field A_μ. The structure of $\phi_a = \alpha\, X_a^{(N)} \otimes \mathbb{1}_n$ suggests to consider the subgroups $SU(N) \times SU(n)$ of $SU(\mathcal{N})$, where $K := SU(n)$ is the commutant of ϕ_a i.e. the maximal subgroup of $SU(\mathcal{N})$ which commutes with all ϕ_a, $a = 1, 2, 3$; this follows from Schur's Lemma. K will turn out to be the effective (low-energy) unbroken four-dimensional gauge group.

One could now proceed in a standard way arguing that $SU(\mathcal{N})$ is spontaneously broken to K since ϕ_a takes a v.e.v. as in (8.15), and elaborate the Higgs mechanism. This is essentially what will be done below, however in a language which is very close to the picture of compactification and KK modes on a sphere in extra dimensions. This is appropriate here, and leads to a description of the low-energy physics of this model as a dimensionally reduced $SU(n)$ gauge theory. Similar calculations have been also presented in section 7.5.

Kaluza-Klein expansion on S^2_N. As in section 7.5, I interpret the $X_a^{(N)}$ as generators of the fuzzy sphere S^2_N and decompose the full four-dimensional $\mathfrak{su}(\mathcal{N})$-valued gauge fields A_μ into spherical harmonics $Y^{lm}(x)$ on the fuzzy sphere S^2_N with coordinates $y^a \sim X_a$:

$$A_\mu = A_\mu(x, X) = \sum_{\substack{0 \le l \le N \\ |m| \le l}} A^{(n)}_{\mu,lm}(x) \otimes Y^{lm(N)}(X)\,. \tag{8.24}$$

The $Y^{lm(N)}$ are by definition irreps under the $SU(2)$ rotations on S^2_N, and form a basis of Hermitian $N \times N$ matrices; for more details see section 6.5. The $A^{(n)}_{\mu,lm}(x)$ turn out to be $\mathfrak{u}(n)$-valued gauge and vector fields on M^4. Using this expansion, I can interpret $A_\mu(x, X)$ as $\mathfrak{u}(n)$-valued functions on $M^4 \times S^2_N$, expanded into the Kaluza-Klein modes (i.e. harmonics) of S^2_N.

The scalar fields ϕ_a with potential (8.23) and vacuum (8.15) should be interpreted as 'covariant coordinates' on S^2_N which describe $U(n)$ Yang-Mills theory on S^2_N. This means that the fluctuations A_a of these covariant coordinates

$$\phi_a = \alpha\, X_a^{(N)} \otimes \mathbb{1}_n + A_a \tag{8.25}$$

should be interpreted as gauge fields on the fuzzy sphere, see appendix B.4. They can

be expanded similarly as

$$A_a = A_a(x,X) = \sum_{\substack{0\leq l\leq N\\ |m|\leq l}} A^{(n)}_{a,lm}(x)\otimes Y^{lm(N)}(X)\,, \tag{8.26}$$

interpreted as functions (or one-form) on $M^4 \times S^2_N$ taking values in $\mathfrak{u}(n)$. One can then interpret $A_M(x,y) = (A_\mu(x,y), A_a(x,y))$ as $\mathfrak{u}(n)$-valued gauge or vector fields on $M^4 \times S^2_N$.

Given this expansion into KK modes, I shall show that only $A_{\mu,00}(y)$ (i.e. the dimensionally reduced gauge field) becomes a massless $\mathfrak{su}(n)$-valued[‡] gauge field in 4D, while all other modes $A_{\mu,lm}(y)$ with $l \geq 1$ constitute a tower of Kaluza-Klein modes with large mass gap, and decouple for low energies. The existence of these KK modes firmly establishes our claim that the model develops dynamically extra dimensions in the form of S^2_N. This geometric interpretation is hence forced upon us, provided the vacuum has the form (8.15). The scalar fields $A_a(x,y)$ will be analysed in a similar way below, and provide no additional massless degrees of freedom in four dimensions. More complicated vacua will have a similar interpretation. Remarkably, our model is fully renormalizable in spite of its higher-dimensional character, in contrast to the commutative case; see also [112].

Computation of the KK masses. To justify these claims, let me compute the masses of the KK modes (8.24). They are induced by the covariant derivatives $\int Tr(D_\mu\phi_a)^2$ in (8.1),

$$\int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a = \int Tr(\partial_\mu\phi_a^\dagger\partial_\mu\phi_a + 2(\partial_\mu\phi_a^\dagger)[A_\mu,\phi_a] + [A_\mu,\phi_a]^\dagger[A_\mu,\phi_a])\,. \tag{8.27}$$

The most general scalar field configuration can be written as

$$\phi_a(x,X) = \alpha(x)X_a^{(N)}\otimes \mathbb{1}_n + A_a(x,X) \tag{8.28}$$

where $A_a(x,X)$ is interpreted as gauge field on the fuzzy sphere S^2_N for each $x\in M^4$. I allow here for a x–dependent $\alpha(x)$ (which could have been absorbed in $A_a(x,X)$), because it is naturally interpreted as the Higgs field responsible for the symmetry breaking $SU(\mathcal{N}) \to SU(n)$. As usual, the last term in (8.27) leads to the mass terms for the gauge fields A_μ in the vacuum $\phi_a(x,X) = \alpha(x)X_a^{(N)}\otimes\mathbb{1}_n$, provided the mixed term which is linear in A_μ does not contribute in the mass term. As I noted in section 7.5, this is achieved by imposing an 'internal' Lorentz gauge $[X_a, A_a] = 0$ at each point x. Then all quadratic contributions of that term vanish, leaving only cubic interaction terms. Indeed, by inserting (8.28) into the term $(\partial_\mu\phi_a^\dagger)[A_\mu,\phi_a]$ in (8.27), I obtain

$$\int TrA_\mu[\phi_a,\partial_\mu\phi_a^\dagger] = \int TrA_\mu\Big\{\alpha(x)[X_a,\partial_\mu A_a(x,X)] + [A_a(x,X),\partial_\mu\alpha(x)\,X_a] + [A_a(x,X),\partial_\mu A_a(x,X)]\Big\}\,,$$

[‡]Note that $A_{\mu,00}(y)$ is traceless, while $A_{\mu,lm}(y)$ is not in general.

which simplifies as

$$\int Tr A_\mu[\phi_a, \partial_\mu \phi_a^\dagger] = \int Tr A_\mu[A_a(x,X), \partial_\mu A_a(x,X)] =: S_{int}\,. \tag{8.29}$$

This contains only cubic interaction terms, which are irrelevant for the computation of the masses. Therefore I can proceed by setting $\phi_a = \alpha X_a^{(N)} \otimes \mathbb{1}_n$ and inserting the expansion (8.24) of A_μ into the last term of (8.27). Since the $Tr[X_a, A_\mu][X_a, A_\mu]$ is the quadratic Casimir on the modes of A_μ which are orthogonal, one obtains

$$\int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a = \int Tr\Big(\partial_\mu\phi_a^\dagger\partial_\mu\phi_a + \sum_{l,m}\alpha^2\, l(l+1)\, A_{\mu,lm}(y)^\dagger A_{\mu,lm}(y)\Big) + S_{int}\,. \tag{8.30}$$

Therefore the four-dimensional $\mathfrak{u}(n)$ gauge fields $A_{\mu,lm}(y)$ acquire a mass

$$m_l^2 = \frac{\alpha^2 g^2}{R^2}\, l(l+1) \tag{8.31}$$

reinserting the parameter R (8.5) which has dimension length. This is as expected for higher KK modes, and determines the radius of the internal S^2 to be

$$r_{S^2} = \frac{\alpha}{g} R \tag{8.32}$$

where $\alpha \approx 1$ according to (8.17). In particular, only $A_\mu(y) \equiv A_{\mu,00}(y)$ survives as a massless four-dimensional $\mathfrak{su}(n)$ gauge field. The low-energy effective action for the gauge sector is then given by

$$S_{LEA} = \int d^4x\, \frac{1}{4g^2}\, \mathrm{Tr}_n\; F_{\mu\nu}^\dagger F_{\mu\nu}\,, \tag{8.33}$$

where $F_{\mu\nu}$ is the field strength of the low-energy $\mathfrak{su}(n)$ gauge fields, dropping all other KK modes whose mass scale is set by $\frac{1}{R}$. For $n = 1$, there is no massless gauge field. However I would find a massless $U(1)$ gauge field if I start with a $U(\mathcal{N})$ gauge theory rather than $SU(\mathcal{N})$.

Scalar sector. As in section 7.5, I expand the most general scalar fields ϕ_a into modes, singling out the coefficient of the 'radial mode' as

$$\phi_a(x) = X_a^{(N)} \otimes (\alpha\mathbb{1}_n + \varphi(x)) + \sum_k A_{a,k}(x) \otimes \varphi_k(x)\,. \tag{8.34}$$

Here $A_{a,k}(x)$ stands for a suitable basis labeled by k of fluctuation modes of gauge fields on S_N^2, and $\varphi(x)$ resp. $\varphi_k(x)$ are $\mathfrak{u}(n)$-valued. The fluctuation modes in the expansion (8.34) have a large mass gap of the order of the KK scale as before. Therefore I can drop these modes for the low-energy sector. However, the field $\varphi(x)$ plays a somewhat special role. It corresponds to fluctuations of the radius of the internal fuzzy

sphere, which is the order parameter responsible for the SSB $SU(\mathcal{N}) \to SU(n)$, and assumes the value $\alpha 1\!\!1_n$ in (8.34). $\varphi(x)$ is therefore the Higgs which acquires a positive mass term in the broken phase, which can be obtained by inserting

$$\phi_a(x) = X_a^{(N)} \otimes (\alpha 1\!\!1_n + \varphi(x))$$

into $V(\phi)$. This mass is dominated by the first term in (8.6) (assuming $a^2 \approx \frac{1}{\tilde{g}^2}$), of order

$$V(\varphi(x)) \approx N\left(a^2 C_2(N)^2 \varphi(x)^2 + O(\varphi^3)\right) \tag{8.35}$$

for large $\mathcal{N}$ and N. The full potential for φ is of course quartic.

In conclusion, the model under discussion indeed behaves like a $U(n)$ gauge theory on $M^4 \times S_N^2$, with the expected tower of KK modes on the fuzzy sphere S_N^2 of radius (8.32). The low-energy effective action is given by the lowest KK mode, which is

$$S_{LEA} = \int d^4x \, \mathrm{Tr}_n \Big[\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + NC_2(N)\, D_\mu\varphi(x) D_\mu\varphi(x) + Na^2 C_2(N)^2 \varphi(x)^2\Big] + S_{int} \tag{8.36}$$

for the $SU(n)$ gauge field $A_\mu(x) \equiv A_{\mu,00}(x)$. In (8.36) I also keep the Higgs field $\varphi(x)$, even though it acquires a large mass

$$m_\varphi^2 = \frac{a^2}{R^2} C_2(N) \tag{8.37}$$

reinserting R.

8.3.2 Type 2 vacuum and $SU(n_1) \times SU(n_2) \times U(1)$ gauge group

For different parameters in the potential, one can obtain a different vacuum, with different low-energy gauge group. Assume now that the vacuum has the form (8.20). The structure of ϕ_a suggests to consider the subgroups $(SU(N_1) \times SU(n_1)) \times (SU(N_2) \times SU(n_2)) \times U(1)$ of $SU(\mathcal{N})$, where

$$K := SU(n_1) \times SU(n_2) \times U(1) \tag{8.38}$$

is the maximal subgroup of $SU(\mathcal{N})$ which commutes with all ϕ_a, $a = 1,2,3$ (this follows from Schur's Lemma). Here the $U(1)$ factor is embedded as

$$\mathfrak{u}(1) \sim \begin{pmatrix} \frac{1}{N_1 n_1} 1\!\!1_{N_1 \times n_1} & \\ & -\frac{1}{N_2 n_2} 1\!\!1_{N_2 \times n_2} \end{pmatrix} \tag{8.39}$$

which is traceless. K will again be the effective (low-energy) four-dimensional gauge group.

I now repeat the above analysis of the KK modes and their effective four-dimensional mass. First, I write

$$A_\mu = \begin{pmatrix} A_\mu^1 & A_\mu^+ \\ A_\mu^- & A_\mu^2 \end{pmatrix} \tag{8.40}$$

according to (8.20), where $(A_\mu^+)^\dagger = -A_\mu^-$. The masses of the gauge bosons are again induced by the last term in (8.27). Consider the term $[\phi_a, A_\mu] = [\alpha_1 X_a^{(N_1)} + \alpha_2 X_a^{(N_2)}, A_\mu]$. For the diagonal fluctuations $A_\mu^{1,2}$, this is simply the adjoint action of $X_a^{(N_1)}$. For the off-diagonal modes $A_\mu^\pm$, I can get some insight by assuming first $\alpha_1 = \alpha_2$. Then the above commutator is $X^{(N_1)} A_\mu^+ - A_\mu^+ X^{(N_2)}$, reflecting the rep. content $A_\mu^+ \in (N_1) \otimes (N_2)$ and $A_\mu^- \in (N_2) \otimes (N_1)$. Assuming $N_1 - N_2 = k > 0$, this implies in particular that there are *no zero modes for the off-diagonal blocks*, rather the lowest angular momentum is k. They can be interpreted as being sections on a monopole bundle with charge k on $S^2_{N_1}$, cf. [151]. The case $\alpha_1 \neq \alpha_2$ requires a more careful analysis as indicated below. In any case, I can again expand A_μ into harmonics,

$$A_\mu = A_\mu(x, X) = \sum_{l,m} \begin{pmatrix} A^1_{\mu,lm}(x)\, Y^{lm(N_1)} & A^+_{\mu,lm}(x)\, Y^{lm(+)} \\ A^-_{\mu,lm}(x)\, Y^{lm(-)} & A^2_{\mu,lm}(x)\, Y^{lm(N_2)} \end{pmatrix}, \tag{8.41}$$

setting $Y^{lm(N)} = 0$ if $l > 2N$. Then the $A^{1,2}_{\mu,lm}(x)$ are $\mathfrak{u}(n_1)$ resp. $\mathfrak{u}(n_2)$-valued gauge resp. vector fields on M^4, while $A^\pm_{\mu,lm}(x)$ are vector fields on M^4 which transform in the bifundamental $(n_1, \overline{n}_2)$ resp. $(n_2, \overline{n}_1)$ of $\mathfrak{u}(n_1) \times \mathfrak{u}(n_2)$.

Now I can compute the masses of these fields. For the diagonal blocks this is the same as in subsection 8.3.1, while the off-diagonal components can be handled by writing

$$Tr([\phi_a, A_\mu][\phi_a, A_\mu]) = 2Tr(\phi_a A_\mu \phi_a A_\mu - \phi_a \phi_a A_\mu A_\mu)\,. \tag{8.42}$$

This gives

$$\begin{aligned} \int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a \;=\; & \int Tr\Big(\partial_\mu\phi_a^\dagger \partial_\mu\phi_a + \sum_{l\geq 0}(m^2_{l,1}\, A^{1\dagger}_{\mu,lm}(x) A^1_{\mu,lm}(x) + m^2_{l,2}\, A^{2\dagger}_{\mu,lm}(x) A^2_{\mu,lm}(x)) \\ & + \sum_{l\geq k} 2m^2_{l;\pm}(A^+_{\mu,lm}(x))^\dagger A^+_{\mu,lm}(x)\Big) \end{aligned} \tag{8.43}$$

similar as in (8.30), with the same gauge choice and omitting cubic interaction terms. In particular, the diagonal modes acquire a KK mass

$$m^2_{l,i} = \frac{\alpha_i^2 g^2}{R^2}\, l(l+1) \tag{8.44}$$

completely analogous to (8.31), while the off-diagonal modes acquire a mass

$$\begin{aligned} m^2_{l;\pm} &= \frac{g^2}{R^2}\,\big(\alpha_1\alpha_2\, l(l+1) + (\alpha_1 - \alpha_2)(X_2^2\alpha_2 - X_1^2\alpha_1)\big) \\ &\approx \frac{g^2}{R^2}\left(l(l+1) + \frac{1}{4}(m_2 - m_1)^2 \; + O(\frac{1}{\mathcal{N}})\right) \end{aligned} \tag{8.45}$$

using (8.17) for $\alpha_i \approx 1$. In particular, all masses are positive.

In conclusion, the gauge fields $A^{1,2}_{\mu,lm}(x)$ have massless components $A^{1,2}_{\mu,00}(x)$ which take values in $\mathfrak{su}(n_i)$ due to the KK-mode $l=0$ (as long as $n_i > 1$), while the bifundamental fields $A^{\pm}_{\mu,lm}(x)$ have no massless components. Note that the mass scales of the diagonal modes (8.44) and the off-diagonal modes (8.45) are essentially the same. This result is similar to the breaking $SU(n_1+n_2) \to SU(n_1) \times SU(n_2) \times U(1)$ through an adjoint Higgs, such as in the $SU(5) \to SU(3) \times SU(2) \times U(1)$ GUT model. In that case, one also obtains massive ('ultraheavy') gauge fields in the bifundamental, whose mass should therefore be identified in our scenario with the mass (8.45) of the off-diagonal massive KK modes $A^{\pm}_{\mu,lm}(x)$. The $U(1)$ factor (8.39) corresponds to the massless components $A^{1,2}_{\mu,00}(x)$ above, which is now present even if $n_i = 1$.

The appropriate interpretation of this vacuum is as a gauge theory on $M^4 \times S^2$, compactified on S^2 which carries a magnetic flux with monopole number $|N_1 - N_2|$. This leads to a low-energy action with gauge group $SU(n_1) \times SU(n_2) \times U(1)$. The existence of a magnetic flux is particularly interesting in the context of fermions, since internal fluxes naturally lead to chiral massless fermions [150]. However this not possible for a minimal set of fermions and their spectrum have to be doubled.

Repeating the analysis of fluctuations for the scalar fields is somewhat messy, and will not be given here. However since the vacuum (8.20) is assumed to be stable, all fluctuations in the ϕ_a will again be massive with mass presumably given by the KK scale, and can therefore be omitted for the low-energy theory. Again, one could interpret the fluctuations $\varphi_{1,2}(y)$ of the radial modes $X_a^{(N_{1,2})} \otimes (\alpha_{1,2} + \varphi_{1,2}(y))$ as low-energy Higgs in analogy to (8.34), responsible for the symmetry breaking $SU(n_1+n_2) \to SU(n_1) \times SU(n_2) \times U(1)$.

8.3.3 Type 3 vacuum and further symmetry breaking

Finally consider a vacuum of the form (8.22). The additional fields φ_a transform in the bifundamental of $SU(n_1) \times SU(n_2)$ and lead to further SSB. Of particular interest is the simplest case

$$\phi_a = \begin{pmatrix} \alpha_1 X_a^{(N_1)} \otimes 1\!\!1_n & \varphi_a \\ -\varphi_a^\dagger & \alpha_2 X_a^{(N_2)} \end{pmatrix} \tag{8.46}$$

corresponding to a would-be gauge group $SU(n) \times U(1)$ according to subsection 8.3.2, which will be broken further. Then $\varphi_a = \begin{pmatrix} \varphi_{a,1} \\ \vdots \\ \varphi_{a,n} \end{pmatrix}$ lives in the fundamental of $SU(n)$ charged under $U(1)$, and transforms as $(N_1) \otimes (N_2)$ under the $SO(3)$ corresponding to the fuzzy sphere(s). As discussed below, by adding a further block, one can get somewhat close to the standard model, with φ_a being a candidate for a low-energy Higgs.

I shall argue that there is indeed such a solution of the equation of motion (8.13) for $|N_1 - N_2| = 2$. Note that since $\varphi_a \in (N_1) \otimes (N_2) = (|N_1 - N_2| + 1) \oplus ... \oplus (N_1 + N_2 - 1)$,

it can transform as a vector under $SO(3)$ only in that case. Hence assume $N_1 = N_2 + 2$, and define $\varphi_a \in (N_1) \otimes (N_2)$ to be the unique component which transform as a vector in the adjoint. One can then show that

$$\phi_a\phi_a = -\begin{pmatrix} \alpha_1^2\, C_2(N_1) \otimes 1\!\!1_{n_1} - \frac{h}{N_1} & 0 \\ 0 & \alpha_2^2\, C_2(N_2) - \frac{h}{N_2} \end{pmatrix} \tag{8.47}$$

where h is a normalisation constant, and

$$\varepsilon_{abc}\phi_b\phi_c \;=\; \begin{pmatrix} (\alpha_1^2 - \frac{g_1}{N_1}\frac{h}{C_2(N_1)})\, X_a^{(N_1)} & (\alpha_1 g_1 + \alpha_2 g_2)\varphi_a \\ -(\alpha_1 g_1 + \alpha_2 g_2)\varphi_a^\dagger & (\alpha_2^2 - \frac{g_2}{N_2}\frac{h}{C_2(N_2)})\, X_a^{(N_2)} \end{pmatrix} \tag{8.48}$$

where $g_1 = \frac{N_1+1}{2}$, $g_2 = -\frac{N_2-1}{2}$. This has the same form as (8.46) but with different parameters. We now have three parameters α_1, α_2, h at our disposal, hence generically this ansatz will provide solutions of the e.o.m. (8.13) which amounts to three equations for the independent blocks. It remains to be investigated whether these are energetically favourable.

The commutant K and further symmetry breaking. To determine the low-energy gauge group i.e. the maximal subgroup K commuting with the solution ϕ_a of type (8.46), consider

$$\begin{aligned}\varepsilon_{abc}\phi_b\phi_c \;&-\; (\alpha_1 g_1 + \alpha_2 g_2)\phi_a = \\ &\begin{pmatrix} (\alpha_1^2 - \alpha_1(\alpha_1 g_1 + \alpha_2 g_2) - \frac{g_1}{N_1}\frac{h}{C_2(N_1)})\, X_a^{(N_1)} & 0 \\ 0 & (\alpha_2^2 - \alpha_2(\alpha_1 g_1 + \alpha_2 g_2) - \frac{g_2}{N_2}\frac{h}{C_2(N_2)})\, X_a^{(N_2)} \end{pmatrix}\end{aligned} \tag{8.49}$$

Unless one of the two coefficients vanishes, this implies that K must commute with (8.49), hence $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ is a subgroup of $SU(n_1) \times SU(n_2) \times U(1)$; here I focus on $SU(n_2) = SU(1)$ being trivial. Then (8.46) implies that $k_1\varphi_a = \varphi_a k_2$ for $k_i \in K_i$, which means that φ_a is an eigenvector of k_1 with eigenvalue k_2. Using a $SU(n_1)$ rotation, I can assume that $\varphi_a^T = (\varphi_{a,1}, 0, \ldots, 0)$. Taking into account the requirement that K is traceless, it follows that $K \cong K_1 \cong SU(n_1 - 1) \subset SU(n_1)$. Therefore the gauge symmetry is broken to $SU(n_1 - 1)$. This can be modified by adding a further block as discussed below.

8.3.4 Towards the standard model

Generalising the above considerations, I can construct a vacuum which is quite close to the standard model. Consider

$$\mathcal{N} = N_1 n_1 + N_2 n_2 + N_3\,, \tag{8.50}$$

for $n_1 = 3$ and $n_2 = 2$. As discussed above, I expect a vacuum of the form

$$\phi_a = \begin{pmatrix} \alpha_1\, X_a^{(N_1)} \otimes 1\!\!1_3 & 0 & 0 \\ 0 & \alpha_2\, X_a^{(N_2)} \otimes 1\!\!1_2 & \varphi_a \\ 0 & -\varphi_a^\dagger & \alpha_3\, X_a^{(N_3)} \end{pmatrix} \tag{8.51}$$

if $\tilde{b} \approx C_2(N_1)$ and $N_1 \approx N_2 = N_3 \pm 2$. Then the unbroken low-energy gauge group would be

$$K = SU(3) \times U(1)_Q \times U(1)_F\,, \tag{8.52}$$

with $U(1)_F$ generated by the traceless generator

$$u(1)_F \sim \begin{pmatrix} \frac{1}{3N_1}\, 1\!\!1_{3N_1} & \\ & -\frac{1}{D}\, 1\!\!1_D \end{pmatrix} \tag{8.53}$$

where $D = 2N_2 + N_3$, and $U(1)_Q$ generated by the traceless generator

$$u(1)_Q \sim \begin{pmatrix} \frac{1}{3N_1}\, 1\!\!1_{3N_1} & & \\ & -\frac{1}{N_2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} 1\!\!1_{N_2} & \\ & & 0 \end{pmatrix}. \tag{8.54}$$

assuming that $\varphi_a^T = (\varphi_{a,1}, 0)$. This is starting to be reminiscent of the standard model, and will be studied in greater detail elsewhere. However, I should recall that the existence of a *vacuum* of this form has not been established at this point.

Relation with CSDR

Let me compare the results of the current approach with the CSDR construction presented in chapter 7. I described there the construction of four-dimensional models starting from gauge theory on $M^4 \times S^2_N$ by imposing CSDR constraints, which were appropriately generalised for the case of fuzzy cosets. The solution of the constraints was boiled down to choosing embeddings ω_a, $a = 1, 2, 3$ of $SU(2)$ into $SU(\mathcal{N})$. which determine the unbroken gauge field as the commutant of ω_a, and the low-energy (unbroken) Higgs by $\varphi_a \sim \omega_a$. Obviously, the vacua solutions (8.15) or (8.20) could be also interpreted as solutions of the Fuzzy-CSDR constraints in [93], provided that appropriate $\omega_a \hookrightarrow SU(2)$ embeddings are chosen.

However, there are important differences. First, the approach described here provides a clear dynamical mechanism which chooses a unique vacuum. This depends crucially on the first term in (8.6), that removes the degeneracy of all possible embeddings of $SU(2)$, which have vanishing field strength F_{ab}. Moreover, it may provide an additional mechanism for further symmetry breaking as discussed in subsection 8.3.3. Another difference is that the starting point in [93] is a six-dimensional gauge theory with some given gauge group, such as $U(1)$. Here the six-dimensional gauge group depends on the parameters of the model.

8.4 Discussion

I have presented a renormalizable four-dimensional $SU(\mathcal{N})$ gauge theory with a suitable multiplet of scalars, which dynamically develops fuzzy extra dimensions that form a fuzzy sphere. The model can then be interpreted as six-dimensional gauge theory, with gauge group and geometry depending on the parameters in the original Lagrangian. I explicitly found the tower of massive Kaluza-Klein modes, consistent with an interpretation as compactified higher-dimensional gauge theory, and determine the effective compactified gauge theory. Depending on the parameters of the model the low-energy gauge group can be $SU(n)$, or broken further e.g. to $SU(n_1) \times SU(n_2) \times U(1)$, with mass scale determined by the extra dimension.

There are many remarkable aspects of this model. First, it provides an extremely simple and geometrical mechanism of dynamically generating extra dimensions. The model is based on a basic lesson from noncommutative gauge theory, namely that noncommutative or fuzzy spaces can be obtained as solutions of matrix models. The mechanism is quite generic, and does not require fine-tuning or supersymmetry. This provides in particular a realisation of the basic ideas of compactification and dimensional reduction within the framework of renormalizable quantum field theory. Moreover, I am essentially considering a large $\mathcal{N}$ gauge theory, which should allow to apply the analytical techniques developed in this context.

One of the main features of the mechanism I presented here is that the effective properties of the model including its geometry depend on the particular parameters of the Lagrangian, which are subject to renormalisation. In particular, the RG flow of these parameters depends on the specific vacuum i.e. geometry, which in turn will depend on the energy scale. For example, it could be that the model assumes a 'type 3' vacuum as discussed in subsection 8.3.3 at low energies, which might be quite close to the standard model. At higher energies, the parameter $\tilde{b}$ (which determines the effective gauge group and which is expected to run quadratically under the RG flow) will change, implying a very different vacuum with different gauge group etc. This suggests a rich and complicated dynamical hierarchy of symmetry breaking, which remains to be elaborated.

In particular, I have shown that the low-energy gauge group is given by $SU(n_1) \times SU(n_2) \times U(1)$ or $SU(n)$, while gauge groups which are products of more than two simple components (apart from $U(1)$) do not seem to occur in this model. The values of n_1 and n_2 are determined dynamically. Moreover, the existence of a magnetic flux in the vacua with non-simple gauge group is very interesting in the context of fermions, since internal fluxes naturally lead to chiral massless fermions [150]. However this not possible for a minimal set of fermions and their spectrum have to be doubled.

There is also an intriguing analogy between the model under discussion and string theory, in the sense that as long as $a = 0$, there are a large number of possible vacua (given by all possible partitions (8.14)) corresponding to compactifications, with no dynamical selection mechanism to choose one from the other. Remarkably this analog of the 'string vacuum problem' is simply solved by adding a term to the action.

Finally I should point out some potential problems or shortcomings of the model. First, the existence of the most interesting vacuum structure of type 3 [(8.46) or (8.51)] has not been yet fully established. Furthermore, the presented results are valid only for a small range of the parameter space of the theory. A complete analysis is expected to give a rich hierarchy of symmetry breaking patterns. Finally, the use of scalar Higgs fields ϕ_a without supersymmetry may seem somewhat problematic due to the strong renormalisation behaviour of scalar fields. This is in some sense consistent with the interpretation as higher-dimensional gauge theory, which would be non-renormalizable in the classical case. Moreover, a large value of the quadratically divergent term $\tilde{b}$ is quite desirable here as explained in subsection 8.2.2, and does not require particular fine-tuning.

Chapter 9

Noncommutative Spacetime and Gravity

In the previous chapters, I followed a rather modest generalisation keeping the continuum characteristic for the ordinary spacetime whereas promoting the extra dimensional space to noncommutative 'manifold'. It is evident that these ideas can be tested also in ordinary spacetime. By following this approach I assume that in the energy regime where the noncommutativity is expected to be valid, the ordinary spacetime cannot be described as a continuum. Elementary cells of length scale μ_P^{-1} (with μ_P denoting the Planck mass) are expected to form out; the ordinary spacetime then is just a limit of this noncommutative 'phase'.

The purpose of the study of noncommutative spacetime is three-fold. First to explore new mathematical ideas and generalisations of the Einstein gravity. Secondly, it has been claimed [92] that the noncommutative generalisation of extra-dimensions can help to avoid quantum divergences generated by operators living over them; the minimum length scale of the noncommutative space elementary cells stand as an ultraviolet cutoff of the otherwise divergent quantities. Of course this is an idea that could be tested in the ordinary spacetime also. Last and probably most importantly, having assumed the extra dimensions in the Planck mass regime to appear noncommutative characteristics would be no profound reason for this behaviour not to be the case in the ordinary dimensions too.

The search for consistent noncommutative deformations of Einstein gravity has been a subject of interest for considerable amount of time. An incomplete list of references is [169–172]. Particularly noteworthy are the gravity theories built on fuzzy spaces. A simple model in two dimensions with Euclidean signature has been developed [107, 173, 174]. More recently, a noncommutative deformation of gravity has been investigated for the case of two-dimensional fuzzy space [175], and some conclusions concerning the emergence of gravity as a macroscopic phenomenon of noncommutativity were made [176,177]. In [178,179] the problem has been studied as deformation of the algebra of diffeomorphisms whereas in [164] a gravitational gravitational was emerged

from a noncommutative gauge theory of a matrix model.

In the following sections, I provide a noncommutative generalisation of Einstein gravity for the approximation of 'small' noncommutativity. For the extra assumption of linear perturbations of both the Poisson structure which defines the algebra and the tetrad of our noncommutative space, a relation between the two notions is found. Linear perturbations of the metric are found to receive corrections controlled by the noncommutative structure of the algebra. Such a result suggests possible relation between gravity and the microscopic noncommutative structure of space.

9.1 Gravity and its correspondence with noncommutative algebras. A Suggestion

It is known fact that to a noncommutative geometry one can associate in various ways a gravitational field. This can be elegantly done [180, 181] in the imaginary-time formalism and perhaps less so [182] in the real-time formalism. Here following [176, 177] I will show a gravitational field is intimately associated with a lack of commutativity of the local coordinates of the space-time structure on which the field is defined. The gravitational field can be described by a moving frame with local components e^{μ}_{α}; the lack of commutativity by a commutator $J^{\mu\nu}$. To a coordinate x^{μ} one associates a conjugate momentum p_{α} and to this couple a commutator

$$[p_{\alpha}, x^{\mu}] = e^{\mu}_{\alpha}\,. \tag{9.1}$$

Let me introduce a set $J^{\mu\nu}$ of elements of an associative algebra $\mathcal{A}$ ('noncommutative space' or 'fuzzy space') defined by commutation relations

$$[x^{\mu}, x^{\nu}] = i\bar{k}J^{\mu\nu}(x^{\sigma})\,. \tag{9.2}$$

The constant $\bar{k}$ is a square of a real number which defines the length scale on which the effects of noncommutativity become important. The $J^{\mu\nu}$ are restricted by Jacobi identities; I shall show below that there are two other requirements which also restrict them.

I suppose the differential calculus over $\mathcal{A}$ to be defined by a frame, a set of one-forms θ^{α} which commute with the elements of the algebra. We assume the derivations dual to these forms to be inner, given by momenta p_{α} as in ordinary quantum mechanics

$$e_{\alpha} = \mathrm{adj}(p_{\alpha})\,. \tag{9.3}$$

Recall that here the momenta p_{α} include a factor $(i\bar{k})^{-1}$. The momenta stand in duality to the position operators by the relation (9.1). However, now consistency relations in the algebra restrict θ^{α} and $J^{\mu\nu}$. Most important thereof is the Leibniz rule which defines differential relations

$$i\bar{k}[p_{\alpha}, J^{\mu\nu}] = [x^{[\mu}, [p_{\alpha}, x^{\nu]}]] = [x^{[\mu}, e^{\nu]}_{\alpha}] \tag{9.4}$$

between the $J^{\mu\nu}$ on the left and the frame components e^μ_α on the right.

Now one can state the relation (9.1) between noncommutativity and gravity more precisely. The right-hand side of this identity defines the gravitational field. The left-hand side must obey Jacobi identities. These identities yield relations between quantum mechanics in the given curved space-time and the noncommutative structure of the algebra. The three aspects of reality then, the curvature of space-time, quantum mechanics and the noncommutative structure are intimately connected.

We resume the various possibilities in a diagram, starting with a classical metric $\tilde{g}_{\mu\nu}$.

$$\begin{array}{ccccc} \tilde{g}_{\mu\nu} & \longrightarrow & \tilde{\theta}^\alpha & \longleftrightarrow & \tilde{\Lambda}^\alpha_\mu \\ & & \downarrow & & \\ g_{\mu\nu} & \longleftarrow & \theta^\alpha & \longleftrightarrow & \Omega(\mathcal{A}) \\ & & \downarrow & & \\ & & J^{\mu\nu} & \longrightarrow & \mathcal{A} \end{array} \tag{9.5}$$

The most important flow of information is from the classical metric $\tilde{g}_{\mu\nu}$ to the commutator $J^{\mu\nu}$, defined in three steps. The first step is to associate to the metric a moving frame $\tilde{\theta}^\alpha$, which can be written in the form $\tilde{\theta}^\alpha = \tilde{\theta}^\alpha_\mu dx^\mu$. The frame is then 'quantised' according to the ordinary rules of quantum mechanics; the dual derivations $\tilde{e}_\alpha$ are replaced by inner derivations $e_\alpha = \mathrm{adj}(p_\alpha)$ of a noncommutative algebra. The commutation relations are defined by the $J^{\mu\nu}$, obtained from the θ^α by solving a differential equation. If the space is flat and the frame is the canonical flat frame then the right-hand side of (9.4) vanishes and it is possible to consistently choose $J^{\mu\nu}$ to be constant or zero. The same is also possible if the e^μ_α happens to belong in the center of algebra $\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$. Note that the map (9.6) is not single valued since any constant $J^{\mu\nu}$ has flat space as inverse image. Conversely a non-trivial noncommutative algebra defined by a non-constant Poisson structure $J^{\mu\nu} = J^{\mu\nu}(x^\sigma)$ generally makes the noncommutative space defined by x^μ curved.

Here, following [176], I define the map

$$J^{\mu\nu} \mapsto \mathrm{Curv}(\theta^\alpha) \tag{9.6}$$

from the Poisson structure $J^{\mu\nu}$ to the curvature of a frame and discuss some interesting conclusions concerning the model I suggest; these were further improved in [177] . The existence of (9.6) allows me to express the Ricci tensor* in terms of $J^{\mu\nu}$. The interest at the moment of this point is limited by the fact that I have no 'equations of motion' for $J^{\mu\nu}$.

*More correctly the noncommutative corrections that the 'would-be' Ricci tensor receives at its commutative limit.

However, since the constraints of our Jacobi system cannot be solved exactly I work in the first order approximation of noncommutativity, controlled by $\kbar$, and in linear perturbation of both flat space and some constant Poisson structure $J_0^{\mu\nu}$; I prove that the former receive corrections from the latter suggesting a deeper relation between noncommutativity and gravity. I denote the order parameters for the perturbation of the metric and the perturbation of the Poisson Structure as ϵ_{GF} and ϵ respectively.

To be more specific let μ be a typical 'large' source mass with 'Schwarzschild radius' $G_N\mu$. Then one has two length scales, determined by respectively the square $G_N\hbar$ of the Planck length and by $\kbar$. The gravitational field is weak if the dimensionless parameter $\epsilon_{GF} = G_N\hbar\mu^2$ is small; the space-time is almost commutative if the dimensionless parameter $\epsilon = \kbar\mu^2$ is small. These two parameters are not necessarily related but since I am interested to allow constant noncommutativity in the flat space limit (c.f. subsection 9.2.3), I shall here assume that they are of the same order of magnitude

$$\epsilon_{GF} \simeq \epsilon \,. \tag{9.7}$$

In section 9.3 I give an example of explicit calculation of the (9.6) for the case of covariant WKB approximation which I mimic here as far as possible. An interesting related conclusion concerns the mode decomposition of the image metric. This was investigated in [177]. It was found there that in the linear approximation there are three modes in all; two dynamical modes of a spin-2 particle plus a scalar mode. They need however not all be present: the graviton will be polarised by certain background noncommutative 'lattice' structure. This leads to the problem of the propagation of the modes in the 'lattice'. Furthermore an energy-momentum for the Poisson structure defined and its eventual contribution of this energy-momentum to the gravitational field equations was studied.

The motivation for considering noncommutative geometry as an 'avatar' of gravity is the belief that it sheds light on the role of the gravitational field as the universal regulator of ultra-violet divergences. Details on these ideas can be found elsewhere [92] and a simple explicit solution in [175]. The model I present is in definite overlap with an interesting recent interpretation [164] of the map (9.6) as a redefinition of the gravitational field in terms of noncommutative electromagnetism.

9.2 The Correspondence

To fix the notation let me briefly recall some elements of the noncommutative frame formalism.

9.2.1 Preliminary formalism

Let me start with a noncommutative $*$-algebra $\mathcal{A}$ generated by four hermitean elements x^μ which satisfy the commutation relations (9.2). Assume that over $\mathcal{A}$ is a differential calculus which is such that the module of one-forms is free and possesses a preferred frame θ^α which commutes,

$$[x^\mu, \theta^\alpha] = 0 \tag{9.8}$$

with the algebra. The metric on this preferred frame, since $[f(x^\mu), \theta^\alpha] = 0$ also, takes the form

$$g = g_{\mu\nu}(dx^\mu \otimes dx^\nu) = g_{\mu\nu}\left[(e^\mu_\alpha \theta^\alpha) \otimes (e^\nu_\beta \theta^\beta)\right] = \underbrace{\left(g_{\mu\nu} e^\mu_\alpha e^\nu_\beta\right)}_{=g_{\alpha\beta}} \left(\theta^\alpha \otimes \theta^\beta\right) \tag{9.9}$$

with the $g_{\alpha\beta}$ to be rescalable in the flat Minkowski metric, $diag(-1,1,1,1)$. The space one obtains in the commutative limit is therefore parallelizable with a global moving frame $\tilde{\theta}^\alpha$ defined to be the commutative limit of θ^α. Then the differential is given by

$$dx^\mu = e^\mu_\alpha \theta^\alpha, \qquad e^\mu_\alpha = e_\alpha x^\mu\,. \tag{9.10}$$

The differential calculus is defined as the largest one consistent with the module structure of the one-forms so constructed. The algebra is defined by a product which is restricted by the matrix of elements $J^{\mu\nu}$; the metric is defined by the matrix of elements e^μ_α. Consistency requirements, essentially determined by Leibniz rules, impose relations between these two matrices which in simple situations allow me to find a one-to-one correspondence between the structure of the algebra and the metric. The input of which I shall make the most use is the Leibniz rule (9.4) which can also be written as relation between one-forms

$$i\bar{k} dJ^{\mu\nu} = [dx^\mu, x^\nu] + [x^\mu, dx^\nu]\,. \tag{9.11}$$

or

$$i\bar{k} e_\alpha J^{\mu\nu} = [e^{[\mu}_\alpha, x^{\nu]}]\,, \tag{9.12}$$

using the definitions of (9.10). One can see here a differential equation for $J^{\mu\nu}$ in terms of e^μ_α. It is obvious that if the space is flat and the frame is the canonical flat frame then the right-hand side of (9.12) vanishes and it is possible to consistently choose $J^{\mu\nu}$

to be constant or zero; the same is also possible if the e^μ_α happens to belong in the center of $\mathcal{A}$ algebra, $\mathcal{Z}(\mathcal{A})$. On the other hand, the noncommutative algebra *must* be defined by a non-constant Poisson structure $J^{\mu\nu} = J^{\mu\nu}(x^\sigma)$, i.e. being *non-trivial*, when e^μ_α does not commute with the elements of the algebra and in addition the $[e^\mu_\alpha, x^\nu]$ matrix has antisymmetric component under $\mu \leftrightarrow \nu$ exchange. Relation (9.12) cannot be solved exactly but in important special cases reduces to a simple differential equation of one variable.

It is important to note that (9.12) or in its equivalent form (9.4) is essentially the Jacobi identity concerning one momentum and two noncommutative 'coordinates'. Keeping the associativity of the noncommutative algebra three other Jacobi identities must be satisfied. The one concerning the x's

$$[x^\mu, [x^\nu, x^\lambda]] + [x^\nu, [x^\lambda, x^\mu]] + [x^\lambda, [x^\mu, x^\nu]] = 0\,, \tag{9.13}$$

constrain additionally our noncommutative system. In subsection 9.2.5 I show that the Jacobi identities concerning two or three momenta are automatically satisfied at least for our working approximations.

In addition, I must insure that the differential is well defined. A necessary condition is that $d[x^\mu, \theta^\alpha] = 0$, from which it follows that the momenta p_α must satisfy the quadratic relation [92]

$$2p_\alpha p_\beta P^{\alpha\beta}{}_{\gamma\delta} - p_\alpha F^\alpha{}_{\gamma\delta} - K_{\alpha\delta} = 0 \tag{9.14}$$

with $P^{\alpha\beta}{}_{\gamma\delta}$, $F^\alpha{}_{\gamma\delta}$, and $K_{\alpha\delta}$ in the center of $\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$. On the other hand, from (9.8) it follows that

$$d[x^\mu, \theta^\alpha] = [dx^\mu, \theta^\alpha] + [x^\mu, d\theta^\alpha] = e^\mu_\beta[\theta^\beta, \theta^\alpha] - \tfrac{1}{2}[x^\mu, C^\alpha{}_{\beta\gamma}]\theta^\beta\theta^\gamma\,, \tag{9.15}$$

where I have introduced the Ricci rotation coefficients

$$d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma\,. \tag{9.16}$$

Therefore I find that multiplication of one-forms must satisfy

$$[\theta^\alpha, \theta^\beta] = \tfrac{1}{2}\theta^\beta_\mu[x^\mu, C^\alpha{}_{\gamma\delta}]\theta^\gamma\theta^\delta\,. \tag{9.17}$$

Using the consistency conditions I obtain that

$$\theta^{[\beta}_\mu[x^\mu, C^{\alpha]}{}_{\gamma\delta}] = 0\,, \tag{9.18}$$

and also that the expression $\theta^{(\alpha}_\mu[x^\mu, C^{\beta)}{}_{\gamma\delta}]$ must be central. Note that these conditions are valid for our working approximations of the next two subsections.

9.2.2 The quasi-commutative approximation

To lowest order in ϵ the partial derivatives are well defined and the approximation, which I shall refer to as the quasi-commutative,

$$[x^\lambda, f] = i\bar{k} J^{\lambda\sigma}\partial_\sigma f + \mathcal{O}\left((i\bar{k})^2\right), \qquad e_\alpha(f) = [p_\alpha, f] = \partial_\alpha f + \mathcal{O}(i\bar{k}). \tag{9.19}$$

is valid. The Leibniz rule (9.12) and the Jacobi identity (9.13) can be written in this approximation as

$$e_\alpha J^{\mu\nu} = \partial_\sigma e^{[\mu}_\alpha J^{\sigma\nu]}\,, \tag{9.20}$$

$$\varepsilon_{\kappa\lambda\mu\nu} J^{\gamma\lambda} e_\gamma J^{\mu\nu} = 0\,. \tag{9.21}$$

I shall refer to these equations including their integrability conditions as the Jacobi equations.

Written in frame components the Jacobi equations become

$$e_\gamma J^{\alpha\beta} - C^{[\alpha}{}_{\gamma\delta} J^{\beta]\delta} = 0\,, \tag{9.22}$$

$$\varepsilon_{\alpha\beta\gamma\delta} J^{\gamma\eta}(e_\eta J^{\alpha\beta} + C^\alpha{}_{\eta\zeta} J^{\beta\zeta}) = 0\,. \tag{9.23}$$

I have used here the expression for the rotation coefficients, valid also in the quasi-commutative approximation:

$$C^\alpha{}_{\beta\gamma} = \theta^\alpha_\mu e_{[\beta} e^\mu_{\gamma]} = -e^\nu_\beta e^\mu_\gamma \partial_{[\nu}\theta^\alpha_{\mu]}\,. \tag{9.24}$$

Taking the antisymmetric part of (9.23) over the α and β indices and taking in account the (9.22) one finds the relation

$$\epsilon_{\alpha\beta\gamma\delta} J^{\alpha\zeta} J^{\beta\eta} C^\gamma{}_{\eta\zeta} = 0\,. \tag{9.25}$$

Now it is possible to solve the (9.22) for the rotation coefficients which are found to be

$$J^{\gamma\eta} e_\eta J^{\alpha\beta} = J^{\alpha\eta} J^{\beta\zeta} C^\gamma{}_{\eta\zeta}\,, \tag{9.26}$$

or, provided J^{-1} exists,

$$C^\alpha{}_{\beta\gamma} = J^{\alpha\eta} e_\eta J^{-1}_{\beta\gamma}\,. \tag{9.27}$$

Note that from general considerations also follows that the rotation coefficients must satisfy the gauge condition

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0\,. \tag{9.28}$$

Equation (9.27) means that in the quasi-classical approximation the linear connection and therefore the curvature can be directly expressed in terms of the commutation relations. This is the content of the map (9.6). Indeed since

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}) \tag{9.29}$$

for the Ricci curvature tensor for example one obtains

$$
\begin{aligned}
2R_{\beta\zeta} = & J_{(\zeta\delta} e^{\alpha} e^{\delta} J^{-1}_{\beta)\alpha} + J^{\alpha\delta} e_{(\zeta} e_{\delta} J^{-1}_{\alpha\beta)} \\
& - J_{(\zeta}{}^{\eta} e^{\alpha} J^{-1}_{\eta\gamma} J^{\gamma\delta} e_{\delta} J^{-1}_{\beta)\alpha} + J^{\alpha\delta} e_{\delta} J^{-1}_{\eta\beta} J^{\eta\gamma} e_{\gamma} J^{-1}_{\alpha\zeta} \\
& + J_{\eta\delta} e^{\delta} J^{-1}_{\beta\alpha} J^{\eta\gamma} e_{\gamma} J^{-1\alpha}_{\zeta} + J^{\alpha\eta} e_{(\zeta} J^{-1}_{\eta\gamma} J^{\gamma\delta} e_{\delta} J^{-1}_{\beta)\alpha} \\
& - \tfrac{1}{2} J_{\zeta\delta} e^{\delta} J^{-1\alpha\eta} J_{\beta\gamma} e^{\gamma} J^{-1}_{\alpha\eta} + J^{\alpha\delta} e_{\delta} J^{-1}_{\alpha\eta} J_{(\zeta\gamma} e^{\gamma} J^{-1\eta}_{\beta)} \,,
\end{aligned} \tag{9.30}
$$

where the ordering on the right-hand side, have been neglected as it gives the corrections of second-order in ϵ, which in our current investigation are omitted. To understand better the relation between the commutator and the curvature in the following section I shall consider a linearisation about a fixed 'ground state'.

9.2.3 Weak field approximation

If I consider the $J^{\mu\nu}$ of the previous sections as the components of a classical field on a curved manifold then in the limit when the manifold becomes flat the 'equations of motion'

$$e_{\alpha} J^{\mu\nu} = 0 \,, \tag{9.31}$$

become Lorentz invariant and obtain as the possible solutions either zero or constant noncommutativity. However, I suppose that as $e^{\lambda}_{\alpha} \to e^{\lambda}_{0\alpha}$ I obtain

$$J^{\mu\nu} \to J^{\mu\nu}_0, \qquad \det J_0 \neq 0 \,, \tag{9.32}$$

which breaks Lorentz Invariance in vacuum; $J^{\mu\nu}_0 = 0$ would be a stronger assumption.

Let me now consider fluctuations around a particular given solution to the problem I have set. I suppose that I have a reference solution comprising a frame $e^{\mu}_{0\alpha} = \delta^{\mu}_{\alpha}$ and a commutation relation $J^{\mu\nu}_0$ which I perturb as

$$J^{\mu\nu} = J^{\mu\nu}_0 + \epsilon\, I^{\mu\nu} \,, \qquad e^{\mu}_{\alpha} = \delta^{\mu}_{\beta}(\delta^{\beta}_{\alpha} + \epsilon_{GF}\, \Lambda^{\beta}_{\alpha}) \,. \tag{9.33}$$

Due to the assumption (9.32) as e^{λ}_{α} reach the constant 'curvature' frame limit it is reasonable to assume $\epsilon \backsim \epsilon_{GF}$ with the gravitational field order parameter to be $\epsilon_{GF} = G_N \hbar \mu^2$. Recall that μ is some typical large gravitational mass and $G_N \hbar$ the square Planck length. In order the second and higher order approximation to be negligible I furthermore assume $\epsilon_{GF} \ll 1$. Then in terms of the unknowns I and Λ the Jacobi and Leibniz constraints (9.12) and (9.13) become respectively

$$\varepsilon_{\lambda\mu\nu\sigma}[x^{\lambda}, I^{\mu\nu}] = 0 \,, \tag{9.34}$$

$$i\hbar e_{\alpha} I^{\mu\nu} = [\Lambda^{\mu}_{\alpha}, x^{\nu}] - [\Lambda^{\nu}_{\alpha}, x^{\mu}] \,. \tag{9.35}$$

In the quasi-commutative approximation [relations (9.19)], the two constraint equations become

$$\varepsilon_{\lambda\mu\nu\sigma} J^{\lambda\sigma}_0 \partial_{\sigma} I^{\mu\nu} = 0 \,, \tag{9.36}$$

$$e_{\alpha} I^{\mu\nu} = \partial_{\sigma} \Lambda^{[\mu}_{\alpha} J^{\sigma\nu]}_0 \,. \tag{9.37}$$

These two equations are the origin of the particularities of the construction I present here, they and the fact that the 'ground-state' value of $J^{\mu\nu}$ is an invertible matrix.

The constraint equations become particularly transparent if one first rewrite them in frame components and then introduce the new unknowns

$$\hat{I}_{\alpha\beta} = J^{-1}_{0\alpha\gamma} J^{-1}_{0\beta\delta} I^{\gamma\delta}\,, \qquad \hat{\Lambda}_{\alpha\beta} = J^{-1}_{0\beta\gamma} \Lambda^{\gamma}_{\alpha}\,. \tag{9.38}$$

Let me also decompose $\hat{\Lambda}$ as the sum

$$\hat{\Lambda}_{\alpha\beta} = \hat{\Lambda}^{+}_{\alpha\beta} + \hat{\Lambda}^{-}_{\alpha\beta} \tag{9.39}$$

of a symmetric and antisymmetric term. The constraints become

$$\partial_\alpha \hat{I}_{\beta\gamma} + \partial_\beta \hat{I}_{\gamma\alpha} + \partial_\gamma \hat{I}_{\alpha\beta} = 0 \tag{9.40}$$

$$e_\alpha \hat{I}_{\beta\gamma} = \partial_{[\beta} \hat{\Lambda}^{+}_{\gamma]\alpha} + \partial_{[\beta} \hat{\Lambda}^{-}_{\gamma]\alpha}\,. \tag{9.41}$$

I introduce

$$\hat{I} = \frac{1}{2} \hat{I}_{\alpha\beta} \theta^\alpha \theta^\beta\,, \qquad \hat{\Lambda}^{-} = \frac{1}{2} \hat{\Lambda}^{-}_{\alpha\beta} \theta^\alpha \theta^\beta\,. \tag{9.42}$$

However, due to the second of (9.19), the (9.40) constraint can be written in a more compact form ('cocycle' condition)

$$\varepsilon^{\alpha\beta\gamma\delta} e_\alpha \hat{I}_{\beta\gamma} = 0, \qquad \text{or} \qquad d\hat{I} = 0\,, \tag{9.43}$$

which is correct up to the first order of my working approximation. Then by taking the (9.41) into account one finds

$$\varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \hat{\Lambda}^{-}_{\beta\gamma} = 0\,, \tag{9.44}$$

since the symmetric part of $\hat{\Lambda}$ has vanishing contribution in the relation. For the approximation of almost flat metric this can be written as

$$\varepsilon^{\alpha\beta\gamma\delta} e_\alpha \hat{\Lambda}^{-}_{\beta\gamma} = 0, \qquad \text{or} \qquad d\hat{\Lambda}^{-} = 0\,, \tag{9.45}$$

which can be rewritten as

$$e_{[\beta} \hat{\Lambda}^{-}_{\gamma]\alpha} = -e_\alpha \hat{\Lambda}^{-}_{\beta\gamma}\,. \tag{9.46}$$

Substituting the last relation in the (9.41) constraint one obtains

$$e_\alpha (\hat{I} + \hat{\Lambda}^{-})_{\beta\gamma} = \partial_{[\beta} \hat{\Lambda}^{+}_{\gamma]\alpha} \tag{9.47}$$

as the final result. This equation has the integrability conditions

$$e_\alpha e_{[\beta} \hat{\Lambda}^{+}_{\gamma]\delta} - e_\delta e_{[\beta} \hat{\Lambda}^{+}_{\gamma]\alpha} = 0\,. \tag{9.48}$$

But the left-hand side is the linearised approximation to the curvature of a metric with components $g_{\mu\nu} + \epsilon\hat{\Lambda}^{+}_{\mu\nu}$. If it vanishes then the perturbation is a derivative; for some one-form A

$$\hat{\Lambda}^{+}_{\beta\gamma} = \frac{1}{2} e_{(\beta} A_{\gamma)} \,. \tag{9.49}$$

Equation (9.47) becomes therefore

$$e_\alpha (\hat{I} + \hat{\Lambda}^{-} - dA)_{\beta\gamma} = 0 \,. \tag{9.50}$$

It follows then that for some two-form c with constant components $c_{\beta\gamma}$

$$\hat{\Lambda}^{-} = -\hat{I} + dA + c \,. \tag{9.51}$$

The remaining constraints are satisfied identically. The most important relation is Equation (9.51) which, in terms of the original 'unhatted' quantities, becomes

$$\Lambda^\alpha_\beta = -J^{-1}_{0\beta\gamma} I^{\alpha\gamma} + J_0^{\alpha\gamma}(c_{\beta\gamma} + e_\gamma A_\beta) \,. \tag{9.52}$$

This condition is much weaker than, but similar to Equation (9.75).

9.2.4 The algebra to geometry map

One can now be more precise about the map (9.6). Let θ^α be a frame which is a small perturbation of a flat frame and let $J^{\alpha\beta}$ be the frame components of a small perturbation of a constant 'background' J_0. Then the map (9.6) is defined by

$$I^{\alpha\beta} \mapsto \Lambda^\alpha_\beta = -J_0^{\alpha\gamma}(\hat{I}_{\gamma\beta} + e_\beta A_\gamma) \,. \tag{9.53}$$

The perturbation on the frame $e^\mu_\alpha = \delta^\mu_\beta(\delta^\beta_\alpha + \epsilon\Lambda^\beta_\alpha)$ engenders a perturbation of the metric. Indeed the bilinearity of the metric implies that the frame components of the metric $g^{\alpha\beta}$ are complex numbers. For the choice of frame I have made and for the working approximation of linear perturbation of flat space I have

$$g^{\alpha\beta} = \eta^{\alpha\beta} - i\epsilon h^{\alpha\beta} \,. \tag{9.54}$$

or in coordinate indices

$$g^{\mu\nu} = g(dx^\mu \otimes dx^\nu) = e^\mu_\alpha e^\nu_\beta g^{\alpha\beta} \,. \tag{9.55}$$

We write $g^{\mu\nu}$ as a sum

$$g^{\mu\nu} = g^{\mu\nu}_{+} + g^{\mu\nu}_{-} \tag{9.56}$$

of symmetric and antisymmetric parts. In the previous section I proved that in lowest order in the noncommutativity in general I have $h^{\alpha\beta} = -h^{\beta\alpha}$ so $g^{\mu\nu}_{+}$ and $g^{\mu\nu}_{-}$ can be decomposed as

$$g^{\mu\nu}_{+} = \tfrac{1}{2}\eta^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta]_{+} - \tfrac{1}{2} i\epsilon h^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta]_{-} \tag{9.57}$$

and

$$g_-^{\mu\nu} = \tfrac{1}{2}\eta^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta]_- - \tfrac{1}{2}i\epsilon h^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta]_+ \,. \tag{9.58}$$

Then for the linear perturbations of the flat space consider one can set

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon g_1^{\mu\nu}, \qquad e^\mu_\alpha = \delta^\mu_\alpha + \epsilon\Lambda^\mu_\alpha \tag{9.59}$$

and the leading order of perturbation in the metric is given as

$$g_1^{\mu\nu} = \eta^{\alpha\beta}\Lambda^{(\mu}_\alpha\delta^{\nu)}_\beta = \Lambda^{(\mu\nu)} \,. \tag{9.60}$$

From (9.53) then

$$g_{1\alpha\beta} = -J_{0(\alpha}{}^\gamma(\hat{I}_{\gamma\beta)} + e_{\beta)}A_\gamma) \tag{9.61}$$

The correction (9.54) will appear only in second order.

The frame itself is given by

$$\theta^\alpha = d(x^\alpha - \epsilon J_0^{\alpha\gamma}A_\gamma) - \epsilon J_0^{\alpha\gamma}\hat{I}_{\gamma\beta}dx^\beta \,. \tag{9.62}$$

Therefore one finds the following expressions

$$d\theta^\alpha = -\epsilon J_0^{\alpha\gamma}e_\delta\hat{I}_{\gamma\beta}dx^\delta dx^\beta = \tfrac{1}{2}\epsilon J_0^{\alpha\delta}e_\delta\hat{I}_{\beta\gamma}dx^\gamma dx^\beta \,, \tag{9.63}$$

$$C^\alpha{}_{\beta\gamma} = \epsilon J_0^{\alpha\delta}e_\delta\hat{I}_{\beta\gamma} \,, \tag{9.64}$$

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2}\epsilon(J_{0[\alpha}{}^\delta e_\delta\hat{I}_{\beta\gamma]} + J_{0\beta}{}^\delta e_\delta\hat{I}_{\alpha\gamma}) \,. \tag{9.65}$$

The torsion obviously vanishes.

Then the linearised Riemann tensor, using (9.65) and the cocycle condition, is given by the expression

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\epsilon\, e^\eta \left(J_{0\eta[\gamma}e_{\delta]}\hat{I}_{\alpha\beta} + J_{0\eta[\alpha}e_{\beta]}\hat{I}_{\gamma\delta}\right) \,. \tag{9.66}$$

For the Ricci curvature I find

$$R_{\beta\gamma} = -\tfrac{1}{2}\epsilon\, e^\zeta \left(J_{0\zeta(\beta}e^\alpha\hat{I}_{\gamma)\alpha} + J_0^\alpha{}_\zeta e_{(\beta}\hat{I}_{\gamma)\alpha}\right) \,. \tag{9.67}$$

One more contraction yields the expression

$$R = -2\epsilon\, J_0{}^{\zeta\alpha}e_\zeta e^\beta\hat{I}_{\alpha\beta} \tag{9.68}$$

for the Ricci scalar. Using again the cocycle condition permits me to write this in the form

$$R = \epsilon\,\Delta\chi \,, \tag{9.69}$$

where the scalar trace component is defined as

$$\chi = J_0^{\alpha\beta}\hat{I}_{\alpha\beta} \,. \tag{9.70}$$

The Ricci scalar is a divergence. Classically it vanishes when the field equations are satisfied.

9.2.5 Phase space

It is obviously the case that in the commutative limit the four coordinate generators tend to the space-time coordinates and the four momenta tend to the conjugate momenta. The eight generators become the coordinates of phase space. For this to be consistent all Jacobi identities must be satisfied, including those with two and three momenta.

I consider first the identities

$$[p_\alpha, [p_\beta, x^\mu]] + [p_\beta, [x^\mu, p_\alpha]] + [x^\mu, [p_\alpha, p_\beta]] = 0\,. \tag{9.71}$$

Using the duality of space-time coordinates and momenta $[p_\alpha, x^\mu] = e^\mu_\alpha$ and the definition $P_{\alpha\beta} = [p_\alpha, p_\beta]$ the above equation is written in a more compact form

$$[x^\mu, P_{\alpha\beta}] = -\Big\{[p_\alpha, e^\mu_\beta] - [p_\beta, e^\mu_\alpha]\Big\}\,, \tag{9.72}$$

which reminds the Leibniz rule (9.11) being in fact its conjugate equivalent. Indeed the (9.71) are satisfied by

$$i\bar{k} P_{\alpha\beta} = \underbrace{(-J^{-1}_{0\alpha\beta} + \epsilon \hat{I}_{\alpha\beta})}_{=-J^{-1}_{\alpha\beta}} + \mathcal{O}(\epsilon^2)\,. \tag{9.73}$$

The remaining identities, involving only the momenta, are then satisfied by virtue of the fact that the two-form $\hat{\Lambda}$ is closed. There is evidence to the fact that this relation is valid to all orders in ϵ.

We also find that

$$[p_\alpha - J^{-1}_{0\alpha\mu} x^\mu, x^\nu] = \delta^\nu_\alpha - J^{-1}_{0\alpha\mu}(J^{\mu\nu}_0 + \epsilon I^{\mu\nu}) + \epsilon \Lambda^\nu_\alpha = \epsilon(\Lambda^\nu_\alpha - J^{-1}_{0\alpha\mu} I^{\mu\nu}) = 0\,. \tag{9.74}$$

For some set of constants c_α therefore, if the center of the algebra is trivial, I can write

$$i\bar{k} p_\alpha = J^{-1}_{0\alpha\mu} x^\mu + c_\alpha\,. \tag{9.75}$$

The 'Fourier transform' is linear.

Let $J^{\mu\alpha}_0$ be an invertible matrix of real numbers. For each such matrix there is an obvious map from the algebra to the geometry given by

$$J^{\mu\nu} \mapsto e^\nu_\alpha = J^{-1}_{0\alpha\mu} J^{\mu\nu}\,. \tag{9.76}$$

For such frames one introduces momenta p_α and find that

$$[p_\alpha, x^\nu] = e^\nu_\alpha = J^{-1}_{0\alpha\mu} J^{\mu\nu} = (i\bar{k})^{-1} J^{-1}_{0\alpha\mu}[x^\mu, x^\nu]\,. \tag{9.77}$$

That is

$$[i\bar{k} p_\alpha - J^{-1}_{0\alpha\mu} x^\mu, x^\nu] = 0\,. \tag{9.78}$$

In conclusion then eq. (9.75) is satisfied. It is reasonable to interpret the results of the previous section as the statement that this condition is stable under small perturbations of the geometry or algebra.

9.2.6 An Example

Consider $(2-d)$-Minkowski space with coordinates (t, x) which satisfy the commutation relations $[t, x] = ht$ and with a geometry encoded in the frame $\theta^1 = t^{-1}dx$, $\theta^0 = t^{-1}dt$. These data describe [92] a noncommutative version of the Lobachevsky plane. The region around the line $t = 1$ can be considered as a vacuum. For the approximations of the previous section to be valid one must rescale t so that in a singular limit the vacuum region becomes the entire space. I can do this by setting

$$t = 1 + ct' \tag{9.79}$$

and consider the limit $c \to 0$. So that the geometry remain invariant I must scale the metric. I do this by rescaling θ^0

$$\theta^0 \mapsto c^{-1}\theta^0 \,. \tag{9.80}$$

The commutation relations become then

$$[t', x] = c^{-1}h + ht' \tag{9.81}$$

and to leading order in c the frame becomes

$$\theta^0 = (1 - ct')dt' \,, \qquad \theta^1 = (1 - ct')dx \,. \tag{9.82}$$

From the definitions (9.33) I find that

$$\begin{aligned} J^{01}_0 &= c^{-1}h \,, & \epsilon I^{01} &= ht' \,, \\ J_{0,01} &= -ch^{-1} \,, & \epsilon\Lambda^\alpha_\beta &= ct'\delta^\alpha_\beta \end{aligned} \tag{9.83}$$

and therefore I obtain the map (9.6) as defined in the previous sections. This example is not quite satisfactory since the cocycle conditions (9.43), (9.45) are vacuous in dimension two.

9.3 The WKB Approximation

Let me now suppose that the algebra $\mathcal{A}$ is a tensor product

$$\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_\omega \tag{9.84}$$

of a ‘slowly-varying’ factor $\mathcal{A}_0$ in which all amplitudes lie and a ‘rapidly-varying’ phase factor which is of order-of-magnitude ϵ so that only functions linear in this factor can appear. The generic element f of the algebra is of the form then

$$f(x^\lambda, \phi) = f_0(x^\lambda) + \epsilon \bar{f}(x^\lambda) e^{i\omega\phi} \tag{9.85}$$

where f_0 and $\bar{f}$ belong to $\mathcal{A}_0$. Because of the condition on ϵ the factor order does not matter and these elements form an algebra. I suppose that both Λ and I belong to $\mathcal{A}_\omega$:

$$\Lambda^\alpha_\beta = \bar{\Lambda}^\alpha_\beta e^{i\omega\phi}, \qquad I^{\alpha\beta} = \bar{I}^{\alpha\beta} e^{i\omega\phi}\,, \tag{9.86}$$

where $\bar{\Lambda}^\alpha_\beta$ and $\bar{I}^{\alpha\beta}$ belong to $\mathcal{A}_0$. Therefore I have also

$$g_1^{\mu\nu} = \bar{g}^{\mu\nu} e^{i\omega\phi}\,. \tag{9.87}$$

Introducing the normal $\xi_\alpha = e_\alpha\phi$ to the surfaces of constant phase and $\eta^\alpha = J_0^{\alpha\beta}\xi_\beta$ I have

$$e_\alpha I_{\beta\gamma} = (i\omega\xi_\alpha \bar{I}_{\beta\gamma} + e_\alpha \bar{I}_{\beta\gamma})e^{i\omega\phi} \tag{9.88}$$

$$e_\alpha \Lambda_{\beta\gamma} = (i\omega\xi_\alpha \bar{\Lambda}_{\beta\gamma} + e_\alpha \bar{\Lambda}_{\beta\gamma})e^{i\omega\phi}\,. \tag{9.89}$$

In the WKB approximation the cocycle condition becomes

$$\xi_\alpha \hat{I}_{\beta\gamma} + \xi_\beta \hat{I}_{\gamma\alpha} + \xi_\gamma \hat{I}_{\alpha\beta} = 0\,. \tag{9.90}$$

I multiply this equation by ξ^α and obtain

$$\xi^2 \hat{I}_{\beta\gamma} + \xi_{[\beta}\hat{I}_{\gamma]\alpha}\xi^\alpha = 0\,. \tag{9.91}$$

If $\xi^2 \neq 0$ then I conclude that

$$\hat{I}_{\beta\gamma} = -\xi^{-2}\xi_{[\beta}\hat{I}_{\gamma]\alpha}\xi^\alpha\,. \tag{9.92}$$

If on the other hand $\xi^2 = 0$ then I conclude that

$$\xi_{[\beta}\hat{I}_{\gamma]\alpha}\xi^\alpha = 0\,, \tag{9.93}$$

which restrict the possible perturbation over the noncommutativite algebra. In terms of the scalar χ I obtain the relation

$$\hat{I}_{\alpha\beta}\eta^\beta = -\tfrac{1}{2}\chi\xi_\alpha\,. \tag{9.94}$$

Using the definition of η I find in the WKB approximation to first order

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2} i\omega\epsilon \left(\eta_{[\alpha}\hat{I}_{\beta\gamma]} + \eta_\beta \hat{I}_{\alpha\gamma}\right), \tag{9.95}$$

$$R_{\alpha\beta\gamma\delta} = -\tfrac{1}{2}\epsilon(i\omega)^2 \left(\eta_{[\gamma}\xi_{\delta]}\hat{I}_{\alpha\beta} - \eta_{[\alpha}\xi_{\beta]}\eta I_{\gamma\delta}\right), \tag{9.96}$$

$$R_{\beta\gamma} = -\tfrac{1}{2}\epsilon(i\omega)^2 \left(\xi_{(\beta}\eta^\alpha - \xi^\alpha\eta_{(\beta}\right)\hat{I}_{\gamma)\alpha}\,, \tag{9.97}$$

$$R = \epsilon(i\omega)^2\chi\xi^2\,. \tag{9.98}$$

In average, the linear-order expressions vanish. One can calculate to second order if I average over several wavelengths. I use the approximations

$$\langle \hat{I}^{\alpha\beta}\rangle = 0\,, \qquad \langle \hat{I}^{\alpha\beta}\hat{I}^{\gamma\delta}\rangle = \frac{1}{2}\hat{\bar{I}}^{\alpha\beta}\hat{\bar{I}}^{\gamma\delta}\,. \tag{9.99}$$

Also as $e_\delta J^{-1}_{\beta\gamma} = -J^{-1}_{\beta\eta} e_\delta J^{\eta\zeta} J^{-1}_{\zeta\gamma}$ I can write $e_\delta J^{-1}_{\beta\gamma} = \epsilon e_\delta \hat{I}_{\beta\gamma}$. Therefore I find expanding (9.30) to second order the expression

$$\langle R_{\beta\gamma}\rangle = \tfrac{1}{2}\epsilon^2(i\omega)^2\Big(\bar{\chi}\xi^\alpha\eta_{(\gamma}\hat{\hat{I}}_{\beta)\alpha} + \tfrac{3}{4}\bar{\chi}^2\xi_\beta\xi_\gamma + \eta^2\hat{\hat{I}}_{\eta\beta}\hat{\hat{I}}^\eta{}_\gamma - \tfrac{1}{2}\eta_\beta\eta_\gamma\hat{\hat{I}}_{\alpha\eta}\hat{\hat{I}}^{\alpha\eta}\Big) \tag{9.100}$$

for the Ricci tensor and the expression

$$\langle R\rangle = \tfrac{1}{8}\epsilon^2(i\omega)^2(2\eta^2\hat{\hat{I}}_{\alpha\beta}\hat{\hat{I}}^{\alpha\beta} + 7\bar{\chi}^2\xi^2) \tag{9.101}$$

for the Ricci scalar.

Note on WKB cohomology

I briefly motivate here the notation used in this section. I introduced the algebra of de Rham forms with a different differential inspired from the WKB approximation. The differential can be introduced for all forms but I give the construction only for the case of two-forms. Let $f_{\alpha\beta}$ be a two-form and define the differential d_ξ of f by the formulae (9.43),(9.45). The interesting point is that the rank of the cohomology module H^2, an elementary form of Spencer cohomology, depends on the norm of ξ. Let c be a 2-cocycle. Then

$$\xi_\alpha c_{\beta\gamma} + \xi_\beta c_{\gamma\alpha} + \xi_\gamma c_{\alpha\beta} = 0. \tag{9.102}$$

Multiplying this by ξ^α one obtains the condition (9.91). There are two possibilities. If $\xi^2 \neq 0$ then it follows immediately that the 2-cocycle is exact. That is, $\mathrm{H}^2 = 0$. If on the other hand $\xi^2 = 0$ then there are cocycles which are not exact. One can think of these as plane-wave solutions to Maxwell's equations. Main result of this section is the dependence of the Riemann tensor uniquely on the cohomology H.

9.4 Discusion

I have derived, following [176, 177], a relation between the structure of an associated algebra as defined by the Poisson structure $J^{\mu\nu}$ of the commutation relations between the generators x^μ on the one hand and the metrics which the algebra can support, that is, which are consistent with the structure of a differential calculus over the algebra on the other. I have expressed this relation as the map (9.6) from the J which defines the algebra to the frame. The essential ingredients in the definition of the map are the Leibniz rules and the assumption (9.8) on the structure of the differential calculus. Although there have been found [183–185] numerous particular examples, there is not yet a systematic discussion of either the range or kernel of the map. I have here to a certain extent alleviated this, but only in the context of perturbation theory around a vacuum and even then, only in the case of a high-frequency wave. A somewhat similar relation has been found [186] in the case of radiative, asymptotically-flat space-times.

One starts with a consistent flat-space solution to the constraints of the algebra and of the geometry, a solution with the unusual property that its momenta and position stand in a relation of simple duality, a consequence of which is the fact that the Fourier-transformation is local. Then she (he) perturbed both structures, the geometric and the algebraic, in a seemingly arbitrary manner, but within the context of linear-perturbation theory and requiring that the constraints remain valid. Finally she (he) completely solve the constraints of the perturbation and exhibit a closed solution as I presented here. I have shown that the degrees-of-freedom or basic modes of the resulting theory of gravity can be put in correspondence with those of the noncommutative structure. As an application of the formalism I have considered a high-frequency perturbation of the metric an assumption which mimic WKB approximation and I have calculated the noncommutative corrections that the Ricci tensor receives both in WKB approximation but also in the more general case of quasi-commutative and weak field approximation.

In [177] the model was investigated further. It turned out that the perturbation of the Poisson structure contributes to the energy-momentum as an additional effective source of the gravitational field. It was stressed there that because of the identification of the gravitational field with the Poisson structure the perturbation of the latter is in fact a reinterpretation of a perturbation of the former and not an extra field. The difference with classical gravity lie in the choice of field equations and in the WKB approximation this amounts only to a modification of the conserved quantity.

Chapter 10

Conclusions

The emergence of the SM of particle physics from a more fundamental theory is still an open problem. Theoretical constructions of describing the fundamental interactions of nature, such as string theory, suggests the existence of extra dimensions, which can be either large or compactified, as I have assumed here. Then, in order to obtain a reasonable low energy model, a suitable scheme of dimensional reduction have to be applied. Among them the CSDR scheme was the first which, making use of higher dimensions, incorporates in a unified manner the gauge and the ad-hoc Higgs sector of the spontaneously broken gauge theories in four dimensions [17–19].

It is worth noting that of particular interest for the construction of fully realistic theories in the framework of CSDR are the following virtues that complemented the original suggestion: (i) The possibility to obtain chiral fermions in four dimensions resulting from vector-like reps of the higher dimensional gauge theory [13, 18], (ii) The possibility to deform the metric of certain non-symmetric coset spaces and thereby obtain more than one scales [18, 52, 187], (iii) The possibility that the popular softly broken supersymmetric four-dimensional chiral gauge theories might have their origin in a higher dimensional supersymmetric theory, which is dimensionally reduced over non-symmetric coset spaces [28–30].

Main objective of the present dissertation was the investigation to which extent applying both CSDR and Wilson flux breaking mechanism one can obtain reasonable low energy models. Applying the CSDR scheme over the six-dimensional compact coset spaces leads to anomaly-free $SO(10)$ and E_6 GUTs. However, their gauge symmetry cannot be broken further towards the SM group structure by an ordinary Higgs mechanism; their four-dimensional scalars belong in the fundamental representation of the $SO(10)$ and E_6 gauge groups. As a way out it has been suggested [20] to additionally apply the Wilson flux breaking mechanism [20–23]. Conclusion of the present research (chapter 4) was that even though the application of the method can lead to reasonable low energy models, none of them has the SM gauge group structure. This point in the direction that either the application of the Hosotani breaking mechanism is not appropriate enough when used with CSDR scheme, or some field content is missing from the higher-dimensional

theory. A possible way out could be the addition of some scalar fields which have to transform in suitable representations of the higher dimensional gauge group. Although that such an assumption is not very appealing for a gauge-Higgs unification scheme it could be imposed by some other more fundamental theory and therefore cannot be omitted from further investigation. Furthermore, relaxing the requirement of the higher dimensional gauge group, i.e. being E_8 and/or the dimensionality of the initial theory, there are more possibilities of reasonable low energy models. It is worth noting that these two assumptions are suggested by the Heterotic string theory but other choices can be also considered [74]. The full study of the problem could be a subject of a future publication. Finally, one could repeat the calculation for the complete problem of the dimensional reduction of an $\mathcal{N}=1$, E_8 supergravity Chapline - Manton action [58, 188].

In chapter 7, I discussed a generalisation of the CSDR scheme by the assumption of providing the extra dimensional spaces with noncommutative characteristics ('fuzzy spaces'). Even though the work presented there was not an original one of the author it was an opportunity to discuss how the ordinary CSDR scheme and its generalised image as applied over Fuzzy spaces differ. Interesting advantages of this generalised scheme of dimensional reduction was (i) the enhancement of the initial gauge symmetry due to the noncommutative characteristics of the internal manifold and (ii) the emergence of renormalizable theories in four dimensions.

In chapter 8, the connection of the Fuzzy-CSDR with the renormalizability of the resulting four-dimensional models was investigated. I examined the inverse problem and assumed a four-dimensional gauge theory with scalar fields and with the most general renormalizable potential. I concluded that fuzzy extra dimensions can be dynamically generated as a vacuum solution of this potential at least for some range of its parameter space. Noteworthy, the initial gauge symmetry was broken by the same vacuum solution into the SM gauge group structure. In [150] the model was generalised to include fermions.

Finally, in chapter 9 the ordinary space-time was also assumed to be a noncommutative 'manifold', but for the energy scale for which this could be possible. As as small contribution on the possible generalisations that the Einstein gravity can have under this assumption, I examined how the curvature and the noncommutative structure of the algebra are related. The problem was studied up to the linear approximation of both notions and resulted in circumstantial evidence that the gravity could be a macroscopic phenomenon of a space-time noncommutative structure which may be assumed in the Planck energy regime.

Appendix A

Tables of low-energy models

A.1 Dimensional reduction over symmetric $6D$ coset spaces

Table A.1: Dimensional reduction over symmetric $6D$ coset spaces. *Particle physics models leading to $SO(10)$ GUTs in four dimensions.*

Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
1a	$E_8 \supset SO(6) \times SO(10)$ $\mathbf{248}=(\mathbf{1},\mathbf{45})+(\mathbf{6},\mathbf{10})+(\mathbf{15},\mathbf{1})$ $+(\mathbf{4},\mathbf{16})+(\overline{\mathbf{4}},\overline{\mathbf{16}})$	$\frac{SO(7)}{SO(6)}$	$SO(10)$	$\mathbf{10}$	$\mathbf{16}_L$ $\mathbf{16}'_L$
2b	$E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4) \supset SU(3) \times U^I(1)$ $E_8 \supset (SU(3) \times U^I(1)) \times SO(10)$ $\mathbf{248}=(\mathbf{1},\mathbf{1})_{(0)}+(\mathbf{1},\mathbf{45})_{(0)}+(\mathbf{8},\mathbf{1})_{(0)}$ $+(\mathbf{3},\mathbf{10})_{(-2)}+(\overline{\mathbf{3}},\mathbf{10})_{(2)}$ $+(\mathbf{3},\mathbf{1})_{(4)}+(\overline{\mathbf{3}},\mathbf{1})_{(-4)}$ $+(\mathbf{1},\mathbf{16})_{(-3)}+(\mathbf{1},\overline{\mathbf{16}})_{(3)}$ $+(\mathbf{3},\mathbf{16})_{(1)}+(\overline{\mathbf{3}},\overline{\mathbf{16}})_{(-1)}$	$\frac{SU(4)}{SU(3)\times U^I(1)}$	$SO(10)\left(\times U^I(1)\right)$	$\mathbf{10}_{(-2)}$ $\mathbf{10}_{(2)}$	$\mathbf{16}_{L(3)}$ $\mathbf{16}_{L(-1)}$ $\mathbf{16}'_{L(3)}$ $\mathbf{16}'_{L(-1)}$

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Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
3d, 6d	$E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4) \supset SU'(3) \times U^{II}(1)$ $SU'(3) \supset SU^a(2) \times U^I(1)$ $Y^{I'} = \frac{b}{3}Y^I + \frac{b}{3}Y^{II}$ $Y^{II'} = \frac{2a}{3}Y^I - \frac{a}{3}Y^{II}$ **or** $E_8 \supset SO(6) \times SO(10)$ $SO(6) \backsim SU(4)$ $SU(4) \supset SU^a(2) \times SU^b(2) \times U^{II}(1)$ $SU^b(2) \supset U^I(1)$ $Y^{I'} = -bY^I$ $Y^{II'} = a\,Y^{II}$ $E_8 \supset (SU^a(2) \times U^{I'}(1) \times U^{II'}(1))$ $\times SO(10)$ $\mathbf{248}=(\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1})_{(0,0)}$ $+(\mathbf{3},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{45})_{(0,0)}$ $+(\mathbf{1},\mathbf{1})_{(-2b,0)} + (\mathbf{1},\mathbf{1})_{(2b,0)}$ $+(\mathbf{2},\mathbf{1})_{(-b,2a)} + (\mathbf{2},\mathbf{1})_{(b,-2a)}$ $+(\mathbf{2},\mathbf{1})_{(-b,-2a)} + (\mathbf{2},\mathbf{1})_{(b,2a)}$ $+(\mathbf{1},\mathbf{10})_{(0,-2a)} + (\mathbf{1},\mathbf{10})_{(0,2a)}$ $+(\mathbf{2},\mathbf{10})_{(b,0)} + (\mathbf{2},\mathbf{10})_{(-b,0)}$ $+(\mathbf{1},\mathbf{16})_{(b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(-b,a)}$ $+(\mathbf{1},\mathbf{16})_{(-b,-a)} + (\mathbf{1},\overline{\mathbf{16}})_{(b,a)}$ $+(\mathbf{2},\mathbf{16})_{(0,a)} + (\mathbf{2},\overline{\mathbf{16}})_{(0,-a)}$	$\left(\frac{SU(3)}{SU^a(2)\times U^{I'}(1)}\right)$ $\times\left(\frac{SU(2)}{U^{II'}(1)}\right)$	$SO(10)$ $\left(\times U^{I'}(1) \times U^{II'}(1)\right)$	$\mathbf{10}_{(0,-2a)}$ $\mathbf{10}_{(0,2a)}$ $\mathbf{10}_{(b,0)}$ $\mathbf{10}_{(-b,0)}$	$\mathbf{16}_{L(b,-a)}$ $\mathbf{16}_{L(-b,-a)}$ $\mathbf{16}_{L(0,a)}$ $\mathbf{16}'_{L(b,-a)}$ $\mathbf{16}'_{L(-b,-a)}$ $\mathbf{16}'_{L(0,a)}$

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Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
4f, 7f	$E_8 \supset SO(6)\times SO(10)$ $SO(6) \backsim SU(4) \supset SU'(3)\times U^{III}(1)$ $SU'(3)\supset SU^a(2)\times U^{II}(1)$ $SU^a(2)\supset U^I(1)$ $Y^{I'} = aY^I + \frac{a}{3}Y^{II} + \frac{a}{3}Y^{III}$ $Y^{II'} = -bY^I + \frac{b}{3}Y^{II} + \frac{b}{3}Y^{III}$ $Y^{III'} = -\frac{2c}{3}Y^I + \frac{c}{3}Y^{III}$ **or** $E_8 \supset SO(6)\times SO(10)$ $SO(6) \backsim SU(4)$ $SU(4)\supset SU^a(2)\times SU^b(2)\times U^{III}(1)$ $SU^a(2)\supset U^{II}(1)$ $SU^b(2)\supset U^I(1)$ $Y^{I'} = a\,Y^I - a\,Y^{II}$ $Y^{II'} = -b\,Y^I - b\,Y^{II}$ $Y^{III'} = -c\,Y^{III}$ $E_8 \supset SO(10)$ $\times U^{I'}(1)\times U^{II'}(1)\times U^{III'}(1)$ $\mathbf{248}=\mathbf{1}_{(0,0,0)}+\mathbf{1}_{(0,0,0)}+\mathbf{1}_{(0,0,0)}$ $+\mathbf{45}_{(0,0,0)}$ $+\mathbf{1}_{(-2a,2b,0)}+\mathbf{1}_{(2a,-2b,0)}$ $+\mathbf{1}_{(-2a,-2b,0)}+\mathbf{1}_{(2a,2b,0)}$ $+\mathbf{1}_{(-2a,0,-2c)}+\mathbf{1}_{(2a,0,2c)}$ $+\mathbf{1}_{(0,-2b,-2c)}+\mathbf{1}_{(0,2b,2c)}$ $+\mathbf{1}_{(-2a,0,2c)}+\mathbf{1}_{(2a,0,-2c)}$ $+\mathbf{1}_{(0,-2b,2c)}+\mathbf{1}_{(0,2b,-2c)}$ $+\mathbf{10}_{(0,0,2c)}+\mathbf{10}_{(0,0,-2c)}$ $+\mathbf{10}_{(0,2b,0)}+\mathbf{10}_{(0,-2b,0)}$ $+\mathbf{10}_{(2a,0,0)}+\mathbf{10}_{(-2a,0,0)}$ $+\mathbf{16}_{(a,b,c)}+\overline{\mathbf{16}}_{(-a,-b,-c)}$ $+\mathbf{16}_{(-a,-b,c)}+\overline{\mathbf{16}}_{(a,b,-c)}$ $+\mathbf{16}_{(-a,b,-c)}+\overline{\mathbf{16}}_{(a,-b,c)}$ $+\mathbf{16}_{(a,-b,-c)}+\overline{\mathbf{16}}_{(-a,b,c)}$	$\left(\frac{SU(2)}{U^{I'}(1)}\right)\times\left(\frac{SU(2)}{U^{II'}(1)}\right)$ $\times\left(\frac{SU(2)}{U^{III'}(1)}\right)$	$SO(10)\left(\times U^{I'}(1)\right.$ $\left.\times U^{II'}(1)\times U^{III'}(1)\right)$	$\mathbf{10}_{(2a,0,0)}$ $\mathbf{10}_{(-2a,0,0)}$ $\mathbf{10}_{(0,2b,0)}$ $\mathbf{10}_{(0,-2b,0)}$ $\mathbf{10}_{(0,0,2c)}$ $\mathbf{10}_{(0,0,-2c)}$	$\mathbf{16}_{L(a,b,c)}$ $\mathbf{16}_{L(-a,-b,c)}$ $\mathbf{16}_{L(-a,b,-c)}$ $\mathbf{16}_{L(a,-b,-c)}$ $\mathbf{16}'_{L(a,b,c)}$ $\mathbf{16}'_{L(-a,-b,c)}$ $\mathbf{16}'_{L(-a,b,-c)}$ $\mathbf{16}'_{L(a,-b,-c)}$
5e	$E_8 \supset SO(6)\times SO(10)$ $SO(6)\backsim SU(4)$ $SU(4)\supset SU^a(2)\times SU^b(2)\times U^I(1)$ $E_8 \supset (SU^a(2)\times SU^b(2)\times U^I(1))$ $\times SO(10)$ $\mathbf{248}=(\mathbf{1},\mathbf{1},\mathbf{1})_{(0)}+(\mathbf{1},\mathbf{1},\mathbf{45})_{(0)}$ $+(\mathbf{3},\mathbf{1},\mathbf{1})_{(0)}+(\mathbf{1},\mathbf{3},\mathbf{1})_{(0)}$ $+(\mathbf{2},\mathbf{2},\mathbf{1})_{(2)}+(\mathbf{2},\mathbf{2},\mathbf{1})_{(-2)}$ $+(\mathbf{1},\mathbf{1},\mathbf{10})_{(2)}+(\mathbf{1},\mathbf{1},\mathbf{10})_{(-2)}$ $+(\mathbf{2},\mathbf{2},\mathbf{10})_{(0)}$ $+(\mathbf{2},\mathbf{1},\mathbf{16})_{(1)}+(\mathbf{2},\mathbf{1},\overline{\mathbf{16}})_{(-1)}$ $+(\mathbf{1},\mathbf{2},\mathbf{16})_{(-1)}+(\mathbf{1},\mathbf{2},\overline{\mathbf{16}})_{(1)}$	$\frac{Sp(4)\times SU(2)}{SU^a(2)\times SU^b(2)\times U^I(1)}$	$SO(10)\left(\times U^I(1)\right)$	$\mathbf{10}_{(0)}$ $\mathbf{10}_{(2)}$ $\mathbf{10}_{(-2)}$	$\mathbf{16}_{L(1)}$ $\mathbf{16}_{L(-1)}$ $\mathbf{16}'_{L(1)}$ $\mathbf{16}'_{L(-1)}$

Table A.2: **Application of Hosotani breaking mechanism on particle physics models which are listed in table A.1.**

Case	Discrete Symmetries	$\mathbf{K}'$	Surviving scalars under $\mathbf{K}'$	Surviving fermions under $\mathbf{K}'$	$\mathbf{K}$	Surviving fermions under $\mathbf{K}$
1a	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$	$(\mathbf{1},\mathbf{1},\mathbf{6})$	$(\mathbf{2},\mathbf{1},\mathbf{4})_L-(\mathbf{2},\mathbf{1},\mathbf{4})'_L$ $(\mathbf{1},\mathbf{2},\mathbf{4})_L+(\mathbf{1},\mathbf{2},\mathbf{4})'_L$	Not Interesting	
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$)	"	$(\mathbf{2},\mathbf{2},\mathbf{1})$	$+\leftrightarrow-$	$SU^{diag}(2)\times SU(4)$	$(\mathbf{2},\mathbf{4})_L+(\mathbf{2},\mathbf{4})'_L$ $(\mathbf{2},\mathbf{4})_L-(\mathbf{2},\mathbf{4})'_L$
2b	W ($\mathbb{1}$)	H Unbroken				
	W × Z ($\mathbb{Z}_4$)	Not Interesting				
3d, 6d	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$ $\Big(\times U^{I'}(1)\times U^{II'}(1)\Big)$	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(b,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(-b,0)}$	$(\mathbf{2},\mathbf{1},\mathbf{4})_{L(b,-a)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(b,-a)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-b,-a)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-b,-a)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(0,a)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(0,a)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(0,a)}+(\mathbf{1},\mathbf{2},\mathbf{4})'_{L(0,a)}$	Not Interesting	
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$)	"	$(\mathbf{2},\mathbf{2},\mathbf{1})_{(b,0)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(-b,0)}$	$+\leftrightarrow-$	$SU^{diag}(2)\times SU(4)$ $\Big(\times U^{I'}(1)\times U^{II'}(1)\Big)$	$(\mathbf{2},\mathbf{4})_{L(b,-a)}+(\mathbf{2},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(b,-a)}-(\mathbf{2},\mathbf{4})'_{L(b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(-b,-a)}+(\mathbf{2},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(-b,-a)}-(\mathbf{2},\mathbf{4})'_{L(-b,-a)}$ $(\mathbf{2},\mathbf{4})_{L(0,a)}+(\mathbf{2},\mathbf{4})'_{L(0,a)}$ $(\mathbf{2},\mathbf{4})_{L(0,a)}-(\mathbf{2},\mathbf{4})'_{L(0,a)}$
4f, 7f	W ($\mathbb{Z}_2$)	$SU^{(i)}(2)\times SU^{(ii)}(2)$ $\times SU(4)$ $\Big(\times U^{I'}(1)\times U^{II'}(1)$ $\times U^{III'}(1)\Big)$	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,2b,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,-2b,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,2c)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,-2c)}$	$-$	Not Interesting	
	W ($\mathbb{Z}_2$)^2	"	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,2c)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{(0,0,-2c)}$	"	"	

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Case	Discrete Symmetries	$\mathbf{K}'$	Surviving scalars under $\mathbf{K}'$	Surviving fermions under $\mathbf{K}'$	$\mathbf{K}$	Surviving fermions under $\mathbf{K}$
	W × Z ($\mathbb{Z}_2 \times \mathbb{Z}_2$)	”	$(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,2b,0)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,-2b,0)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,0,2c)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{(0,0,-2c)}$	”	”	
5e	W ($\mathbb{Z}_2$)	$SU^{(i)}(2) \times SU^{(ii)}(2)$ $\times SU(4)$ $\Big(\times U(1)\Big)$	$(\mathbf{1},\mathbf{1},\mathbf{6})_{(0)}$	$(\mathbf{2},\mathbf{1},\mathbf{4})_{L(1)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(1)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(1)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1)} - (\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1)}$ $(\mathbf{1},\mathbf{2},\mathbf{4})_{L(-1)} + (\mathbf{1},\mathbf{2},\mathbf{4})'_{L(-1)}$	Not Interesting	
	W ($\mathbb{Z}_2)^2$	”	$(\mathbf{2},\mathbf{2},\mathbf{1})_{(0)}$	”	$SU^{diag}(2) \times SU(4)$ $\Big(\times U(1)\Big)$	$(\mathbf{2},\mathbf{4})_{L(1)} - (\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(1)} + (\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)} - (\mathbf{2},\mathbf{4})'_{L(-1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)} + (\mathbf{2},\mathbf{4})'_{L(-1)}$
	W × Z ($\mathbb{Z}_2 \times \mathbb{Z}_2$)	”	”	$+ \leftrightarrow -$	”	$(\mathbf{2},\mathbf{4})_{L(1)} + (\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(1)} - (\mathbf{2},\mathbf{4})'_{L(1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)} + (\mathbf{2},\mathbf{4})'_{L(-1)}$ $(\mathbf{2},\mathbf{4})_{L(-1)} - (\mathbf{2},\mathbf{4})'_{L(-1)}$

A.2 Dimensional reduction over non-symmetric $6D$ coset spaces

Table A.3: Dimensional reduction over symmetric $6D$ coset spaces. *Particle physics models leading to E_6 GUTs in four dimensions.*

Case	Embedding	6D Coset Space	H	Surviving scalars under H	Surviving fermions under H
2a′	$E_8 \supset SU'(3) \times E_6$ $E_8 \supset SU'(3) \times E_6$ $\mathbf{248}=(\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{78})$ $+(\mathbf{3},\mathbf{27})+(\bar{\mathbf{3}},\overline{\mathbf{27}})$	$\frac{G_2}{SU'(3)}$	E_6	$\mathbf{27}$ $\overline{\mathbf{27}}$	$\mathbf{78}$ $\mathbf{27}_L$ $\mathbf{27}'_L$
3b′	$E_8 \supset SU'(3) \times E_6$ $SU'(3) \supset SU^a(2) \times U^I(1)$ $Y^{I'} = -Y^I$ $E_8 \supset SU^a(2) \times U^{I'}(1) \times E_6$ $\mathbf{248}=(\mathbf{1},\mathbf{1})_{(0)}+(\mathbf{1},\mathbf{78})_{(0)}$ $+(\mathbf{3},\mathbf{1})_{(0)}$ $+(\mathbf{2},\mathbf{1})_{(-3)}+(\mathbf{2},\mathbf{1})_{(3)}$ $+(\mathbf{1},\mathbf{27})_{(2)}+(\mathbf{1},\overline{\mathbf{27}})_{(-2)}$ $+(\mathbf{2},\mathbf{27})_{(-1)}+(\mathbf{2},\overline{\mathbf{27}})_{(1)}$	$\left(\frac{Sp(4)}{SU^a(2)\times U^I(1)}\right)_{nonmax}$	$E_6\left(\times U^{I'}(1)\right)$	$\mathbf{27}_{(2)}$ $\mathbf{27}_{(-1)}$ $\overline{\mathbf{27}}_{(-2)}$ $\overline{\mathbf{27}}_{(1)}$	$\mathbf{1}_{(0)}$ $\mathbf{78}_{(0)}$ $\mathbf{27}_{L(2)}$ $\mathbf{27}_{L(-1)}$ $\mathbf{27}'_{L(2)}$ $\mathbf{27}'_{L(-1)}$
4c′	$E_8 \supset SU'(3) \times E_6$ $SU'(3) \supset SU^a(2) \times U^{II}(1)$ $SU^a(2) \supset U^I(1)$ $E_8 \supset E_6 \times U^I(1) \times U^{II}(1)$ $\mathbf{248}=\mathbf{1}_{(0,0)}+\mathbf{1}_{(0,0)}$ $+\mathbf{78}_{(0)}$ $+\mathbf{1}_{(-2,0)}+\mathbf{1}_{(2,0)}$ $+\mathbf{1}_{(-1,3)}+\mathbf{1}_{(1,-3)}$ $+\mathbf{1}_{(1,3)}+\mathbf{1}_{(-1,-3)}$ $+\mathbf{27}_{(0,-2)}+\overline{\mathbf{27}}_{(0,2)}$ $+\mathbf{27}_{(-1,1)}+\overline{\mathbf{27}}_{(1,-1)}$ $+\mathbf{27}_{(1,1)}+\overline{\mathbf{27}}_{(-1,-1)}$	$\frac{SU(3)}{U^I(1)\times U^{II}(1)}$	$E_6\left(\times U^I(1) \times U^{II}(1)\right)$	$\mathbf{27}_{(0,-2)}$ $\mathbf{27}_{(-1,1)}$ $\mathbf{27}_{(1,1)}$ $\overline{\mathbf{27}}_{(0,2)}$ $\overline{\mathbf{27}}_{(1,-1)}$ $\overline{\mathbf{27}}_{(-1,-1)}$ $(a{=}0,c{=}-2)$ $(b{=}-1,d{=}1)$	$\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,0)}$ $\mathbf{78}_{(0,0)}$ $\mathbf{27}_{L(0,-2)}$ $\mathbf{27}_{L(-1,1)}$ $\mathbf{27}_{L(1,1)}$ $\mathbf{27}'_{L(0,-2)}$ $\mathbf{27}'_{L(-1,1)}$ $\mathbf{27}'_{L(1,1)}$

Table A.4: Application of Hosotani breaking mechanism on particle physics models which are listed in table A.3. *The surviving fields are calculated for the embeddings $\mathbb{Z}_2 \hookrightarrow E_6$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \hookrightarrow E_6$, discussed in subsections 3.3.1 and 3.3.2*

Case	Discrete Symmetries	K′	Surviving fermions under K′
2a′	W $(\mathbb{Z}_2)$ [*embedding* (**1**)]	$SO(10) \times U(1)$	$\mathbf{1}_{(0)}$ $\mathbf{45}_{(0)}$ $\mathbf{1}_{L(-4)}+\mathbf{1}'_{L(-4)}$ $\mathbf{10}_{L(-2)}+\mathbf{10}'_{L(-2)}$ $\mathbf{16}_{L(1)}-\mathbf{16}'_{L(1)}$

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Case	Discrete Symmetries	$\mathbf{K}'$	Surviving fermions under $\mathbf{K}'$
	W ($\mathbb{Z}_2$) [*embeddings* (**2**), (**3**)]	$SU(2)\times SU(6)$	$(\mathbf{3},\mathbf{1})$ $(\mathbf{1},\mathbf{35})$ $(\mathbf{1},\mathbf{15})_L+(\mathbf{1},\mathbf{15})'_L$ $(\mathbf{2},\overline{\mathbf{6}})_L-(\mathbf{2},\overline{\mathbf{6}})'_L$
3b′	W ($\mathbb{Z}_2$) [*embedding* (**1**)]	$SO(10)\times U(1)\left(\times U^I(1)\right)$	$\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,0)}$ $\mathbf{45}_{(0,0)}$ $\mathbf{1}_{L(-4,2)}+\mathbf{1}'_{L(-4,2)}$ $\mathbf{10}_{L(-2,2)}+\mathbf{10}'_{L(-2,2)}$ $\mathbf{16}_{L(1,2)}-\mathbf{16}'_{L(1,2)}$ $\mathbf{1}_{L(-4,-1)}+\mathbf{1}'_{L(-4,-1)}$ $\mathbf{10}_{L(-2,-1)}+\mathbf{10}'_{L(-2,-1)}$ $\mathbf{16}_{L(1,-1)}-\mathbf{16}'_{L(1,-1)}$
	W ($\mathbb{Z}_2$) [*embeddings* (**2**), (**3**)]	$SU(2)\times SU(6)\left(\times U^I(1)\right)$	$(\mathbf{1},\mathbf{1})_{(0)}$ $(\mathbf{3},\mathbf{1})_{(0)}$ $(\mathbf{1},\mathbf{35})_{(0)}$ $(\mathbf{1},\mathbf{15})_{L(2)}+(\mathbf{1},\mathbf{15})'_{L(2)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(2)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(2)}$ $(\mathbf{1},\mathbf{15})_{L(-1)}+(\mathbf{1},\mathbf{15})'_{L(-1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(-1)}-(\mathbf{2},\overline{\mathbf{6}})'_{L(-1)}$
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$) [*embedding* (**2′**)]	$SU(2)\times SU(6)\left(\times U^I(1)\right)$	$(\mathbf{1},\mathbf{1})_{(0)}$ $(\mathbf{3},\mathbf{1})_{(0)}$ $(\mathbf{1},\mathbf{35})_{(0)}$ $(\mathbf{1},\mathbf{15})_{L(2)}-(\mathbf{1},\mathbf{15})'_{L(2)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(2)}+(\mathbf{2},\overline{\mathbf{6}})'_{L(2)}$ $(\mathbf{1},\mathbf{15})_{L(-1)}-(\mathbf{1},\mathbf{15})'_{L(-1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(-1)}+(\mathbf{2},\overline{\mathbf{6}})'_{L(-1)}$
	W × Z ($\mathbb{Z}_2\times\mathbb{Z}_2$) [*embedding* (**3′**)]	$SU^{(i)}(2)\times SU^{(ii)}(2)\times SU(4)\times U(1)\left(\times U^I(1)\right)$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)}$ $(\mathbf{3},\mathbf{1},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{3},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{15})_{(0,0)}$ $(\mathbf{1},\mathbf{1},\mathbf{1})_{L(4,2)}+(\mathbf{1},\mathbf{1},\mathbf{1})'_{L(4,2)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{L(2,2)}+(\mathbf{2},\mathbf{2},\mathbf{1})'_{L(2,2)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{L(-2,2)}+(\mathbf{1},\mathbf{1},\mathbf{6})'_{L(-2,2)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1,2)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1,2)}$ $(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,2)}-(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,2)}$ $(\mathbf{1},\mathbf{1},\mathbf{1})_{L(4,-1)}+(\mathbf{1},\mathbf{1},\mathbf{1})'_{L(4,-1)}$ $(\mathbf{2},\mathbf{2},\mathbf{1})_{L(2,-1)}+(\mathbf{2},\mathbf{2},\mathbf{1})'_{L(2,-1)}$ $(\mathbf{1},\mathbf{1},\mathbf{6})_{L(-2,-1)}+(\mathbf{1},\mathbf{1},\mathbf{6})'_{L(-2,-1)}$ $(\mathbf{2},\mathbf{1},\mathbf{4})_{L(-1,-1)}-(\mathbf{2},\mathbf{1},\mathbf{4})'_{L(-1,-1)}$ $(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,-1)}-(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{L(1,-1)}$

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Case	Discrete Symmetries	$\mathbf{K}'$	Surviving fermions under $\mathbf{K}'$
$\mathbf{4c}'$	W $(\mathbb{Z}_2)$ [*embedding* $(\mathbf{1})$]	$SO(10) \times U(1) \left(\times U^I(1) \times U^{II}(1) \right)$	$\mathbf{1}_{(0,0,0)}$ $\mathbf{1}_{(0,0,0)}$ $\mathbf{1}_{(0,0,0)}$ $\mathbf{45}_{(0,0,0)}$ and $\mathbf{1}_{L(-4,0,-2)} + \mathbf{1}'_{L(-4,0,-2)}$ $\mathbf{10}_{L(-2,0,-2)} + \mathbf{10}'_{L(-2,0,-2)}$ $\mathbf{16}_{L(1,0,-2)} - \mathbf{16}'_{L(1,0,-2)}$ or $\mathbf{1}_{L(-4,-1,1)} + \mathbf{1}'_{L(-4,-1,1)}$ $\mathbf{10}_{L(-2,-1,1)} + \mathbf{10}'_{L(-2,-1,1)}$ $\mathbf{16}_{L(1,-1,1)} - \mathbf{16}'_{L(1,-1,1)}$ or $\mathbf{1}_{L(-4,1,1)} + \mathbf{1}'_{L(-4,1,1)}$ $\mathbf{10}_{L(-2,1,1)} + \mathbf{10}'_{L(-2,1,1)}$ 1 $\mathbf{16}_{L(1,1,1)} - \mathbf{16}'_{L(1,1,1)}$
	W $(\mathbb{Z}_2)$ [*embeddings* $(\mathbf{2})$, $(\mathbf{3})$]	$SU(2) \times SU(6) \left(\times U^I(1) \times U^{II}(1) \right)$	$\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,0)}$ $(\mathbf{3},\mathbf{1})_{(0,0)}$ $(\mathbf{1},\mathbf{35})_{(0,0)}$ and $(\mathbf{1},\mathbf{15})_{L(0,-2)} + (\mathbf{1},\mathbf{15})'_{L(0,-2)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(0,-2)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(0,-2)}$ or $(\mathbf{1},\mathbf{15})_{L(-1,1)} + (\mathbf{1},\mathbf{15})'_{L(-1,1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(-1,1)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(-1,1)}$ or $(\mathbf{1},\mathbf{15})_{L(1,1)} + (\mathbf{1},\mathbf{15})'_{L(1,1)}$ $(\mathbf{2},\overline{\mathbf{6}})_{L(1,1)} - (\mathbf{2},\overline{\mathbf{6}})'_{L(1,1)}$

Appendix B

Matrix Models

B.1 Gauge theory on the fuzzy sphere (multi-matrix model)

Here I briefly review the construction of Yang-Mills gauge theory on S^2_N as multi-matrix model [151, 165, 166]. Consider the action

$$S \;=\; \frac{4\pi}{\mathcal{N}}\,\mathrm{Tr}\Big(a^2(\phi_a\phi_a + C_2(N))^2 + \frac{1}{\tilde{g}^2}\,F^\dagger_{ab}F_{ab}\Big) \tag{B.1}$$

where $\phi_a = -\phi_a^\dagger$ is an antihermitean $\mathcal{N}\times\mathcal{N}$ matrix, and define*

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon^c{}_{ab}\phi_c\,, \tag{B.2}$$

as I did in section 6.5. This action is invariant under the $U(\mathcal{N})$ 'gauge' symmetry acting as

$$\phi_a \to U^{-1}\phi_a U$$

A priori, I do not assume any underlying geometry, which arises dynamically. I claim that it describes $U(n)$ Yang-Mills gauge theory on the fuzzy sphere S^2_N, assuming that $\mathcal{N} = Nn$.

To see this, note first that the action is positive definite, with global minimum $S = 0$ for the 'vacuum' solution

$$\phi_a = X_a^{(N)} \otimes 1\!\!1_n \tag{B.3}$$

where $X_a \equiv X_a^{(N)}$ are the generators of the N- dimensional irrep. of $SU(2)$. This is a first indication that the model 'dynamically generates' its own geometry, which is the fuzzy sphere S^2_N. In any case, it is natural to write a general field ϕ_a in the form

$$\phi_a = (X_a \otimes 1\!\!1_n) + A_a, \tag{B.4}$$

*This can indeed be seen as components of the two-form $F = dA + AA$.

and to consider $A_a = \sum_\alpha A_a^\alpha(x)\,\mathcal{T}^\alpha$ as functions $A_a^\alpha(x) = -A_a^\alpha(x)^\dagger$ on the fuzzy sphere S_N^2, taking value in $u(n)$ with generators T_α. The gauge transformation then takes the form

$$\begin{aligned} A_a &\rightarrow U^{-1} A_a U + U^{-1}[X_a, U] \\ &= U^{-1} A_a U - i U^{-1} J_a U, \end{aligned} \tag{B.5}$$

which is the transformation rule of a $U(n)$ gauge field. The field strength becomes

$$\begin{aligned} F_{ab} &= [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - \varepsilon^c{}_{ab} A_c \\ &= -i J_a A_b + i J_b A_a + [A_a, A_b] - \varepsilon^c{}_{ab} A_c. \end{aligned} \tag{B.6}$$

This look like the field strength of a nonabelian $U(n)$ gauge field, with the caveat that one has three degrees of freedom rather than two. To solve this puzzle, consider again the action, writing it in the form

$$S = \frac{4\pi}{\mathcal{N}} \operatorname{Tr}\Big(a^2 \varphi^2 + \frac{1}{\tilde{g}^2} F_{ab}^\dagger F_{ab}\Big), \tag{B.7}$$

where I introduce the scalar field

$$\varphi := \phi_a \phi_a + C_2(N) = X_a A_a + A_a X_a + A_a A_a. \tag{B.8}$$

Since only configurations where φ and F_{ab} are small will significantly contribute to the action, it follows that

$$x_a A_a + A_a x_a = O(\frac{\varphi}{N}) \tag{B.9}$$

is small. This means that A_a is tangential in the (commutative) large N limit, and two tangential gauge degrees of freedom† survive. Equivalently, one can use the scalar field $\phi = N\varphi$, which would acquire a mass of order N and decouple from the theory.

I have thus established that the matrix model (B.1) is indeed a fuzzy version of pure $U(n)$ Yang-Mills theory on the sphere, in the sense that it reduces to the commutative model in the large N limit. Without the term $(\phi_a \phi_a + C_2(N))^2$, the scalar field corresponding to the radial component of A_a no longer decouples and leads to a different model.

The main message to be remembered is the fact that the matrix model (B.1) without any further geometrical assumptions dynamically generates the space S_N^2, and the fluctuations turn out to be gauge fields governed by a $U(n)$ Yang-Mills action. Furthermore, the vacuum has no flat directions‡

†To recover the familiar form of gauge theory, one needs to rotate the components locally by $\frac{\pi}{2}$ using the complex structure of S^2. A more elegant way to establish the interpretation as Yang-Mills action can be given using differential forms on S_N^2.

‡The excitations turn out to be monopoles as expected [151].

B.2 Stability of the 'fuzzy sphere solution'

To establish stability of the vacua (8.15), (8.20) one should work out the spectrum of excitations around this solution and check whether there are flat or unstable modes. This is a formidable task in general, and here I only consider the simplest case of the irreducible vacuum (8.15) for the case $\tilde{b} = C_2(N)$ and $d = 0$ here. Once I have established that all fluctuation modes have strictly positive eigenvalues, the same will hold in a neighbourhood of this point in the moduli space of couplings $(a, b, d, \tilde{g}, g_6)$.

An intuitive way to see this is by noting that the potential $V(\phi_a)$ can be interpreted as Yang-Mills theory on S^2_N with gauge group $U(n)$. Since the sphere is compact, I expect that all fluctuations around the vacuum $\phi_a = X_a^{(N)} \otimes 1\!\!1_n$ have positive energy. I fix $n = 1$ for simplicity. Thus I write

$$\phi_a = X_a + A_a(x) \tag{B.10}$$

where $A_a(x)$ is expanded into a suitable basis of harmonics of S^2_N, which I should find. It turns out that a convenient way of doing this is to consider the antihermitean $2N \times 2N$ matrix [151]

$$\Phi = -\frac{i}{2} + \phi_a \sigma_a = \Phi_0 + A \tag{B.11}$$

which satisfies

$$\Phi^2 = \phi_a \phi_a - \frac{1}{4} + \frac{i}{2} \varepsilon_{abc} F_{bc} \sigma_a \, . \tag{B.12}$$

Thus $\Phi^2 = -\frac{N^2}{4}$ for $A = 0$, and in general I have

$$\tilde{S}_{YM} := \mathrm{Tr}(\Phi^2 + \tilde{b} + \frac{1}{4})^2 = \mathrm{Tr}\Big((\phi_a \phi_a + \tilde{b})^2 + F_{ab}^\dagger F_{ab}\Big). \tag{B.13}$$

The following maps turn out to be useful:

$$\mathcal{D}(f) := i\{\Phi_0, f\}, \qquad \mathcal{J}(f) := [\Phi_0, f] \tag{B.14}$$

for any matrix f. The maps $\mathcal{D}$ and $\mathcal{J}$ satisfy

$$\mathcal{J}\mathcal{D} = \mathcal{D}\mathcal{J} = i[\Phi_0^2, .], \qquad \mathcal{D}^2 - \mathcal{J}^2 = -2\{\Phi_0^2, .\}, \tag{B.15}$$

which for the vacuum under consideration become

$$\mathcal{J}\mathcal{D} = \mathcal{D}\mathcal{J} = 0, \qquad \mathcal{D}^2 - \mathcal{J}^2 = N^2, \qquad \mathcal{J}^3 = -N^2 \, \mathcal{J}. \tag{B.16}$$

Note also that

$$\mathcal{J}^2(f) = [\phi_a, [\phi_a, f]] =: -\Delta f \tag{B.17}$$

is the Laplacian, with eigenvalues $\Delta f_l = l(l+1) f_l$ (for the vacuum).

It turns out that the following is a natural basis of fluctuation modes:

$$\begin{aligned} \delta\Phi^{(1)} &= A_a^{(1)}\sigma_a = \mathcal{D}(f) - f, \\ \delta\Phi^{(2)} &= A_a^{(2)}\sigma_a = \mathcal{J}^2(f') - \mathcal{J}^2(f')_0 = \mathcal{J}^2(f') + \Delta f' \\ \delta\Phi^{(g)} &= A_a^{(g)}\sigma_a = \mathcal{J}(f'') \end{aligned} \tag{B.18}$$

for antihermitean $N \times N$ matrices f, f', f'', which will be expanded into orthonormal modes $f = \sum f_{l,m} Y_{lm}$. Using orthogonality it is enough to consider these modes separately, i.e. $f = f_l = -f_l^\dagger$ with $\mathrm{Tr}(f_l^\dagger f_l) = 1$. One can show that these modes form a complete set of fluctuations around Φ_0 (for the vacuum). Here $A_{(g)}$ corresponds to gauge transformations, which I will omit from now on. Using

$$\mathrm{Tr}(f\mathcal{J}(g)) = -\mathrm{Tr}(\mathcal{J}(f)g),, \qquad \mathrm{Tr}(f\mathcal{D}(g)) = \mathrm{Tr}f(\mathcal{D}(f)g) \tag{B.19}$$

I can now compute the inner product matrix $\mathrm{Tr}A^{(i)}A^{(j)}$:

$$\begin{aligned} \mathrm{Tr}(A^{(1)}A^{(1)}) &= \mathrm{Tr}(((N^2-1)f - \Delta(f))g), \\ \mathrm{Tr}(A^{(1)}A^{(2)}) &= \mathrm{Tr}(\Delta(f)g), \\ \mathrm{Tr}(A^{(2)}A^{(2)}) &= \mathrm{Tr}((N^2\Delta(f) - \Delta^2 f))g). \end{aligned} \tag{B.20}$$

It is convenient to introduce the matrix of normalizations for the modes $A^{(i)}$,

$$G_{ij} \equiv \mathrm{Tr}((A^{(i)})^\dagger A^{(j)}) = \begin{pmatrix} (N^2-1) - \Delta, & \Delta \\ \Delta, & N^2\Delta - \Delta^2 \end{pmatrix} \tag{B.21}$$

which is positive definite except for the zero mode $l = 0$ where $A^{(2)}$ is not defined.

We can now expand the action (B.1) up to second order in these fluctuations. Since $F_{ab} = 0$ and $(\phi_a\phi_a + \tilde{b}) = 0$ for the vacuum, I have[§]

$$\delta^2 S_{YM} = \mathrm{Tr}\Big(-\frac{1}{\tilde{g}^2}\,\delta F_{ab}\delta F_{ab} + a^2\delta(\phi_a\phi_a)\delta(\phi_b\phi_b)\Big). \tag{B.22}$$

If $a^2 \geq \frac{1}{\tilde{g}^2}$, this can be written as

$$\begin{aligned} \delta^2 S_{YM} &= \mathrm{Tr}\Big(\frac{1}{\tilde{g}^2}\,(-\delta F_{ab}\delta F_{ab} + a^2\delta(\phi_a\phi_a)\delta(\phi_a\phi_a)) + (a^2 - \frac{1}{\tilde{g}^2})\delta(\phi_a\phi_a)\delta(\phi_a\phi_a)\Big) \\ &= \mathrm{Tr}\Big(\frac{1}{\tilde{g}^2}\,\delta\Phi^2\delta\Phi^2 + (a^2 - \frac{1}{\tilde{g}^2})\delta(\phi_a\phi_a)\delta(\phi_a\phi_a)\Big) \end{aligned} \tag{B.23}$$

and similarly for $a^2 < \frac{1}{\tilde{g}^2}$. It is therefore enough to show that

$$\delta^2 \tilde{S}_{YM} = \mathrm{Tr}(\delta\Phi^2\delta\Phi^2) = \mathrm{Tr}(-\delta^{(i)}F_{ab}\delta^{(j)}F_{ab} + \delta^{(i)}(\phi\cdot\phi)\delta^{(j)}(\phi\cdot\phi)) \tag{B.24}$$

[§]Note that $\delta\mathrm{Tr}(\phi\cdot\phi) = 0$ except for the zero mode $A_0^{(1)}$ with $l = 0$ where $\delta^{(1)}\mathrm{Tr}(\phi\cdot\phi) \neq 0$, as follows from (B.25). This mode corresponds to fluctuations of the radius, which will be discussed separately.

has a finite gap in the excitation spectrum. This spectrum can be computed efficiently as follows: note first

$$\begin{aligned}
\delta^{(1)}\Phi^2 &= -i\mathcal{D}^2(f) + i\mathcal{D}(f) = -i\mathcal{J}^2(f) + i\mathcal{D}(f) - iN^2 f, \\
\delta^{(2)}\Phi^2 &= -i\mathcal{D}(\Delta f), \\
\delta^{(g)}\Phi^2 &= -i\mathcal{D}\mathcal{J}(f) = [\Phi_0^2, f] = 0
\end{aligned} \tag{B.25}$$

for the vacuum. One then finds

$$\begin{aligned}
\mathrm{Tr}(\delta^{(1)}(\Phi^2)\delta^{(1)}(\Phi^2)) &= -\mathrm{Tr}(f)((-(N^2+1)\Delta + (N^2-1)N^2)g), \\
\mathrm{Tr}(\delta^{(1)}(\Phi^2)\delta^{(2)}(\Phi^2)) &= -\mathrm{Tr}(f)(\Delta^2)(g), \\
\mathrm{Tr}(\delta^{(2)}(\Phi^2)\delta^{(2)}(\Phi^2)) &= -\mathrm{Tr}(g)(-\Delta^3 + N^2\Delta^2)g).
\end{aligned} \tag{B.26}$$

Noting that the antihermitean modes satisfy $\mathrm{Tr}(f_l f_l) = -1$, this gives

$$\delta^2 \tilde{S}_{YM} = \begin{pmatrix} -(N^2+1)\Delta + N^4 - N^2, & \Delta^2 \\ \Delta^2, & -\Delta^3 + N^2\Delta^2 \end{pmatrix} = GT \tag{B.27}$$

where the last equality defines T. The eigenvalues of T are found to be N^2 and Δ. These eigenvalues coincide[¶] with the spectrum of the fluctuations of $\tilde{S}_{YM}$. In particular, all modes with $l > 0$ have positive mass. The $l = 0$ mode

$$A_0^{(1)} = \mathcal{D}(f_0) - f_0 = (2i\Phi_0 - 1)f_0 = 2if_0\,\sigma_a\phi_a \tag{B.28}$$

requires special treatment, and corresponds precisely to the fluctuations of the normalization α, i.e. the radius of the sphere. We have shown explicitly in (7.46) that this $\alpha = \alpha(y)$ has a positive mass. Therefore I conclude that all modes have positive mass, and there is no flat or unstable direction. This establishes the stability of this vacuum.

The more general case $\tilde{b} = C_2(N) + \epsilon$ with $\alpha \neq 1$ could be analysed with the same methods, which however will not be presented in this dissertation. For the reducible vacuum (8.20) or (8.22) the analysis is more complicated, and will not be carried out here.

B.3 Stability of ‘type-1’ and ‘type-2’ vacua

To verify the stability of ‘type-1’ and ‘type-2’ vacua solution I used the following Mathematica code:

[¶]To see this, assume that I use an orthonormal basis $A^o_{(i)}$ instead of the basis (B.18), i.e. $A = b_1 A^o_{(1)} + b_2 A^o_{(2)}$. Then I can write $G = g^T g$ and $b_i = g_{ij} a_j$. Thus (B.27) becomes $a^T\, GT\, a = b^T\, g\, T\, g^{-1} b$, and the eigenvalues of $g\, T\, g^{-1}$ coincide with those of T, which therefore gives the masses.

```
<< LinearAlgebra`MatrixManipulation`
<< Algebra`ReIm`

tensorProductIdentity[Amatrix_, n_] :=
  Module[{aux, result},
    aux = Amatrix;
    If[n > 1,
      Do[
        result =
          BlockMatrix[{{aux,ZeroMatrix[Length[aux], Length[Amatrix]]},
                      {ZeroMatrix[Length[Amatrix], Length[aux]], Amatrix}}];
        aux = result,
        {i, 1, n - 1}
      ],
      If[n == 1,
        result = Amatrix,
        Print[
          "Unexpected value for (Dim of Identity tensor product term)=n.\\
           VACconstruction[Np>=2,n>=1] is expected."];
          Abort[]
      ]
    ];
    Return[result]
  ]

VACIconstruction[Np_,n_] :=
  Module[{J0, Jplus, Jminus, x, X, j, m, alpha, phi, vac},
    Clear[phi];
    Im[jt] ^= 0;
    j = (Np - 1)/2;
    Jminus = Table[
              KroneckerDelta[mp,mm + 1](((j + mp)!(j - mm)!)/((j + mm)!(j - mp)!))^(1/2),
              {mp, -j, j}, {mm, -j, j}
            ];
    Jplus = Table[
             KroneckerDelta[mp,mm - 1](((j + mm)!(j - mp)!)/((j + mp)!(j - mm)!))^(1/2),
             {mp, -j, j}, {mm, -j, j}
           ];
    J0 = Table[KroneckerDelta[mp, m] (m), {mp, -j, j}, {m, -j, j}];
    x[1] = (I/2)*(Jplus + Jminus);
    x[2] = (1/2)*(Jplus - Jminus);
    x[3] = (I) *J0;
    m = 2*(j - jt);
    alpha =1 - m/(2 * j);
    Do[
      If[Np >= 2 ,
         X[k] = x[k],
         Print["\<Unexpected value for (Dim of rep)=Np.\\
                  VACconstruction[Np>=2,n>=1] is expected.\>"];
         Abort[]
      ];
      phi[k] = Simplify[alpha*tensorProductIdentity[X[k], n]],
      {k, 1, 3}
    ];
   vac = {phi[1], phi[2], phi[3]}
 ]
```

```
A1reps[j_] :=
  Module[{J0, Jminus, Jplus, x, xlist},
     Jminus = Table[
                KroneckerDelta[mp, mm + 1] (((j + mp)!(j - mm)!)/((j + mm)!(j - mp)!))^(1/2),
                {mp, -j, j}, {mm, -j, j}
               ];
     Jplus = Table[
               KroneckerDelta[mp, mm - 1](((j + mm)!(j - mp)!)/((j + mp)!(j - mm)!))^(1/2),
               {mp, -j, j}, {mm, -j, j}
              ];
     J0 = Table[KroneckerDelta[mp, m] (m), {mp,-j, j}, {m, -j,j}];
     x[1] = (I/2)*(Jplus + Jminus);
     x[2] = (1/2)* (Jplus - Jminus);
     x[3] = (I)* J0;
     xlist = {x[1], x[2], x[3]}
 ]

VACIIconstruction[Np1_, n1_, n2_] :=
   Module[{x1, x2, X1, X2, j1, m1, alpha1, Np2, j2, m2, alpha2, vac},
    Clear[phi];
    Im[jt] ^= 0;
    j1 = (Np1 - 1)/2;
    x1 = A1reps[j1];
    m1 = 2(j1 - jt);
    alpha1 = 1 - m1/(2*j1);
    Np2 = Np1 + 1;
    j2 = (Np2 - 1)/2;
    x2 = A1reps[j2];
    m2 = 2(j2 - jt);
    alpha2 = 1 - m2/(2*j2);
    Do[If[Np1 >= 2 ,
         X1[k] = x1[[k]],
         Print["Unexpected value for (Dim of rep)=Np. VACconstruction[Np>=2,n>=1] is expected."];
         Abort[]
        ];
      If[Np2 >= 2 ,
        X2[k] = x2[[k]],
        Print["Unexpected value for (Dim of rep)=Np. VACconstruction[Np>=2,n>=1] is expected."];
        Abort[]
      ];
      phi1[k] = Simplify[alpha1*tensorProductIdentity[X1[k], n1]];
      phi2[k] = Simplify[alpha2*tensorProductIdentity[X2[k], n2]];
      phi[k] =
        BlockMatrix[{{phi1[k],ZeroMatrix[Length[ phi1[k] ], Length[ phi2[k] ] ]},
                    {ZeroMatrix[Length[phi2[k]], Length[phi1[k]]], phi2[k]}}],
      {k, 1, 3}
    ];
    vac = {phi[1], phi[2], phi[3]}
  ]


VACIperturbation[Np_, n_] :=
  Module[{perturbedMatrix, NBig, f},
    NBig = Np*n;
    Clear[perturbedMatrix];
    perturbedMatrix = ZeroMatrix[NBig];
    Do[
      f = Random[Real, {-1, +1}];
      perturbedMatrix = ReplacePart[perturbedMatrix, f, {{i, j}, {j, i}}],
      {i, 1, NBig}, {j, 1, NBig}
      ];
    Return[- I *perturbedMatrix]
   ]
```

```
VACIIperturbation[Np1_, n1_, n2_] :=
  Module[{perturbedMatrix, Np2, NBig, f},
    Np2 = Np1 + 1;
    NBig = Np1*n1 + Np2*n2;
    Clear[perturbedMatrix];
    perturbedMatrix = ZeroMatrix[NBig];
    Do[
      f = Random[Real, {-1, +1}];
      perturbedMatrix = ReplacePart[perturbedMatrix, f, {{i, j}, {j, i}}],
      {i, 1, NBig}, {j, 1, NBig}
      ];
    Return[- I *perturbedMatrix]
   ]

perturbedVACI[Np_, n_] :=
  Module[{},
    Clear[e];
    e={e1,e2,e3};
    Im[e1] ^= 0;
    Im[e2] ^= 0;
    Im[e3] ^= 0;
    Table[
      VACIconstruction[Np, n][[k]] + e[[k]] *VACIperturbation[Np, n],
      {k, 1, 3}
      ]
    ]

perturbedVACII[Np1_, n1_, n2_] :=
  Module[{},
    Clear[e];
    e={e1,e2,e3};
    Im[e1] ^= 0;
    Im[e2] ^= 0;
    Im[e3] ^= 0;
    Table[
      VACIIconstruction[Np1, n1, n2][[k]] +
        e[[k]] *VACIIperturbation[Np1, n1, n2],
      {k, 1, 3}
      ]
    ]

perturbedFieldStrengthI[Np_, n_] :=
  Module[{perturbedphi},
    perturbedphi = perturbedVACI[Np, n];
    Table[
      (Dot[perturbedphi[[a]], perturbedphi[[b]] ] -
            Dot[perturbedphi[[b]], perturbedphi[[a]]]) -
        Sum[Signature[{a, b, c}]perturbedphi[[c]], {c, 1, 3}],
      {a, 1, 3}, {b, 1, 3}
      ]
    ]

perturbedFieldStrengthII[Np1_, n1_, n2_] :=
  Module[{perturbedphi},
    perturbedphi = perturbedVACII[Np1, n1, n2];
    Table[
      (Dot[perturbedphi[[a]], perturbedphi[[b]] ] -
            Dot[perturbedphi[[b]], perturbedphi[[a]]]) -
        Sum[Signature[{a, b, c}]perturbedphi[[c]], {c, 1, 3}],
      {a, 1, 3}, {b, 1, 3}
      ]
    ]
```

```
perturbedPotentialI[Np_, n_, InvgtSqr_] :=
  Module[{perturbedF},
    perturbedF = perturbedFieldStrengthI[Np, n];
    Collect[
      Simplify[
        Tr[
          Sum[-InvgtSqr*Dot[perturbedF[[a, b]], perturbedF[[a, b]] ] ,
            {a, 1, 3}, {b, 1, 3}]
          ]
        ],
      e]
    ]

perturbedPotentialII[Np1_, n1_, n2_, InvgtSqr_] :=
  Module[{perturbedF},
    perturbedF = perturbedFieldStrengthII[Np1, n1, n2];
    Collect[
      Simplify[
        Tr[
          Sum[-InvgtSqr*Dot[perturbedF[[a, b]], perturbedF[[a, b]] ] ,
            {a, 1, 3}, {b, 1, 3}]
          ]
        ],
      e]
    ]

potentialI[Np_, n_, InvgtSqr_, jmin_] :=
  Module[{perturbedPotential, perturbedPotentialRe, perturbedPotentialIm,
    potentialVacAux},
    Clear[e1,e2,e3,jt];
    Im[e1] ^= 0;
    Im[e2] ^= 0;
    Im[e3] ^= 0;
    perturbedPotential = perturbedPotentialI[Np, n, InvgtSqr];
    perturbedPotentialIm = Im[perturbedPotential];
    perturbedPotentialRe = Expand[Re[perturbedPotential]];

    potential1[e1_,e2_,e3_,jt_] = perturbedPotentialRe;

    potentialVacAux = perturbedPotentialRe /. {e1 -> 0, e2 -> 0, e3 ->0};
    potentialVac1[jt_] := potentialVacAux;

    Print["Perturbed Potential"];
    Print[perturbedPotential];
    Print[" ********************************************"];
    Print["Imaginary Part of Perturbed Potential"];
    Print[perturbedPotentialIm];
    Print[" ********************************************"];
    Print["Real Part of Perturbed Potential"];
    Print[perturbedPotentialRe];
    Print["*********************************************"];
    Print["N=", Np*n, "," , " ", "Np=", Np, ",", " ", "n=", n ];
    DeleteFile[{"potentialI","potentialVACI"}];
    Save["potentialI",potential1];
    Save["potentialVACI",potentialVac1]
  ]
```

```
PotentialII[Np1_, n1_, n2_, InvgtSqr_, jmin_] :=
  Module[{perturbedPotential, perturbedPotentialRe, perturbedPotentialIm,
    potentialVacAux},
    Clear[e1,e2,e3,jt];
    Im[e1] ^= 0;
    Im[e2] ^= 0;
    Im[e3] ^= 0;
    perturbedPotential = perturbedPotentialII[Np1, n1, n2, InvgtSqr];
    perturbedPotentialIm = Im[perturbedPotential];
    perturbedPotentialRe = Expand[Re[perturbedPotential]];

    potential2[e1_,e2_,e3_,jt_] = perturbedPotentialRe;

    potentialVacAux = perturbedPotentialRe /. {e1 -> 0, e2 -> 0, e3 ->0};
    potentialVac2[jt_] := potentialVacAux;

    Print["Perturbed Potential"];
    Print[perturbedPotential];
    Print["*********************************************"];
    Print["Imaginary Part of Perturbed Potential"];
    Print[perturbedPotentialIm];
    Print["*********************************************"];
    Print["Real Part of Perturbed Potential"];
    Print[perturbedPotentialRe];
    Print["*********************************************"];
    Print["N=", Np1*n1 + (Np1 + 1)*n2, "," , " ", "N1=", Np1, ",", " ", "n1=",
          n1, ",", " ", "N2=", Np1 + 1, ",", " ", "n2=", n2 ];
    DeleteFile[{"potentialII","potentialVACII"}];
    Save["potentialII",potential2];
    Save["potentialVACII",potentialVac2]
 ]
```

I run the code on the HET-cluster of the Physics Department of N.T.U.A. with the choices:

type-1

$N = 39$ and $n = 3$
being the $\mathcal{N} = Nn = 117$ partition of $\mathcal{N} \times \mathcal{N}$ matrices.

type-2

$N_1 = 23$, $n_1 = 3$ and $N_2 = N_1 + 1$, $n_2 = 2$
being the $\mathcal{N} = n_1 N_1 + n_2 N_2 = 117$ partition of $\mathcal{N} \times \mathcal{N}$ matrices.

These are within the approximation of large matrices assumed in chapter 8 up to second decimal digit precision. We have also choose $\frac{1}{\tilde{g}^2} = 0.9$ for the coupling constant of the $F_{ab}F^{ab}$ term of the matrix model potential presented in chapter 8. Finally the $\tilde{j}$ has been chosen to be in the $N_1 < \tilde{N} < N_2 = N_1 + 1$ interval.

The following results prove our arguments of the stability of 'type-1' and 'type-2' solutions

type-1

```
potential1[e1$_, e2$_, e3$_, jt$_] = 8310.196242138307*e1^2 + 0.*e1^4 +
     8171.922774302844*e2^2 + 675171.794769941*e1^2*e2^2 + 0.*e2^4 +
     0.*e1*e2*e3 + 8245.841526584754*e3^2 + 642124.806455661*e1^2*e3^2 +
     648257.0741220384*e2^2*e3^2 + 0.*e3^4 + 8.730939070642787*e1*jt +
     0.*e1^3*jt + 0.*e1^2*e2*jt + 5.302009621272134*e1*e2^2*jt +
     7.694136324951049*e3*jt - 419.89704482079856*e1*e3*jt -
     210.06603550339628*e1^2*e3*jt + 0.*e2*e3*jt - 4000.7618414361873*e2^2*e3*
      jt - 0.4532475695330902*e1*e3^2*jt + 0.*e2*e3^2*jt + 0.*e3^3*jt +
     221.68421052631604*jt^2 - 1.3785693269434898*e1*jt^2 +
     11703.294608077817*e1^2*jt^2 + 0.*e1*e2*jt^2 +
     11446.737752030098*e2^2*jt^2 - 1.2148636302556433*e3*jt^2 -
     140.10470181632246*e1*e3*jt^2 + 0.*e2*e3*jt^2 +
     11299.83308575747*e3^2*jt^2 - 23.335180055401562*jt^3 +
     0.0483708535769628*e1*jt^3 + 0.04262679404404693*e3*jt^3 +
     0.6140836856684655*jt^4

Attributes[e1$] = {Temporary}

Attributes[e2$] = {Temporary}

Attributes[e3$] = {Temporary}

Attributes[jt$] = {Temporary}

Im[e1] ^= 0

Im[e2] ^= 0

Im[e3] ^= 0

Im[jt] ^= 0


Local Minima

{4686.55,

{e1 -> 2.44977x10^{-6}  , e2 -> 1.14775x10^{-19}   , e3 -> 2.23931x10^{-6}  }}
```

type-2

```
potential2[e1$_, e2$_, e3$_, jt$_] = 8239.417547028384*e1^2 + 0.*e1^4 +
     8203.264441042475*e2^2 + 630723.2670988459*e1^2*e2^2 + 0.*e2^4 +
     0.*e1*e2*e3 + 8127.235004958384*e3^2 + 644909.1812379633*e1^2*e3^2 +
     605788.2695466514*e2^2*e3^2 + 0.*e3^4 + 13.585420318861358*e1*jt +
     0.*e1^3*jt + 0.*e1^2*e2*jt + 978.7956924200289*e1*e2^2*jt +
     11.315311245671497*e3*jt + 549.6356339064939*e1*e3*jt -
     1629.9495936550218*e1^2*e3*jt + 0.*e2*e3*jt + 265.54923446855196*e2^2*e3*
      jt + 1156.5670778145934*e1*e3^2*jt + 0.*e2*e3^2*jt + 0.*e3^3*jt +
     229.4039525691708*jt^2 - 3.8127881735926827*e1*jt^2 +
     12036.411465663265*e1^2*jt^2 + 0.*e1*e2*jt^2 +
     11971.194007631077*e2^2*jt^2 - 3.0508573620735935*e3*jt^2 -
     19.438795293342935*e1*e3*jt^2 + 0.*e2*e3*jt^2 +
     11587.559130487403*e3^2*jt^2 - 40.96741395741206*jt^3 +
     0.23732001554284943*e1*jt^3 + 0.18286354102580768*e3*jt^3 +
     1.829877102902616*jt^4

Attributes[e1$] = {Temporary}

Attributes[e2$] = {Temporary}

Attributes[e3$] = {Temporary}

Attributes[jt$] = {Temporary}

Im[e1] ^= 0

Im[e2] ^= 0

Im[e3] ^= 0

Im[jt] ^= 0


Local Minima

{13.6354,
{e1 -> -2.40332x10^{-6}  , e2 -> -1.60605x10^{-20}   , e3 -> -3.19194x10^{-7}  }}
```

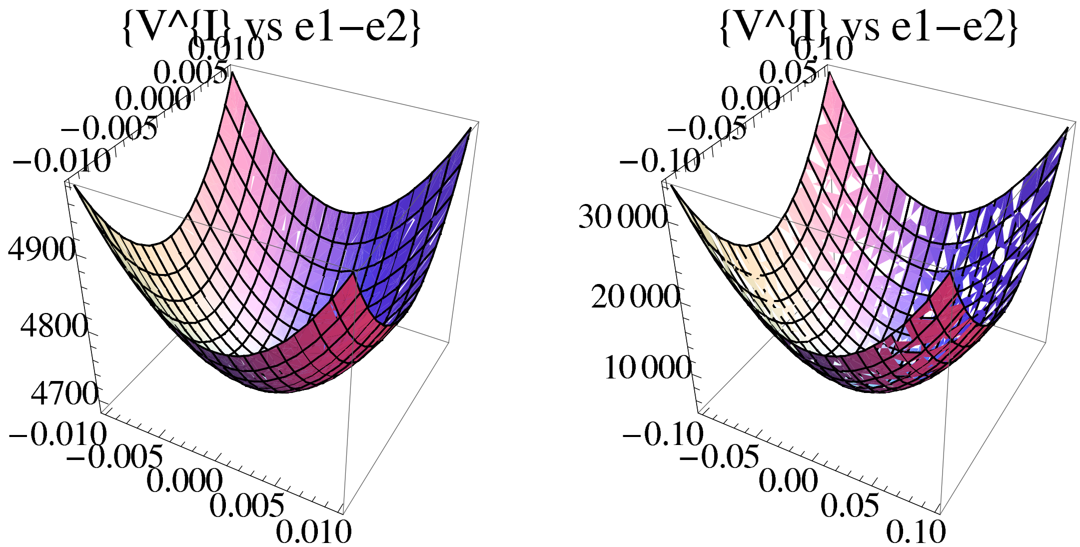

Figure B.1: *'type-1' Potential*

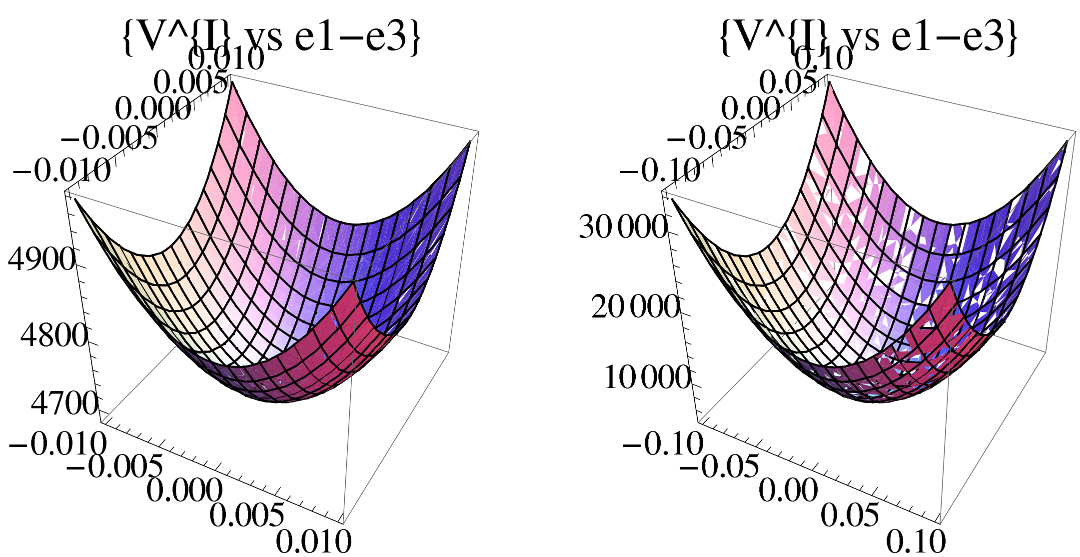

Figure B.2: *'type-1' Potential*

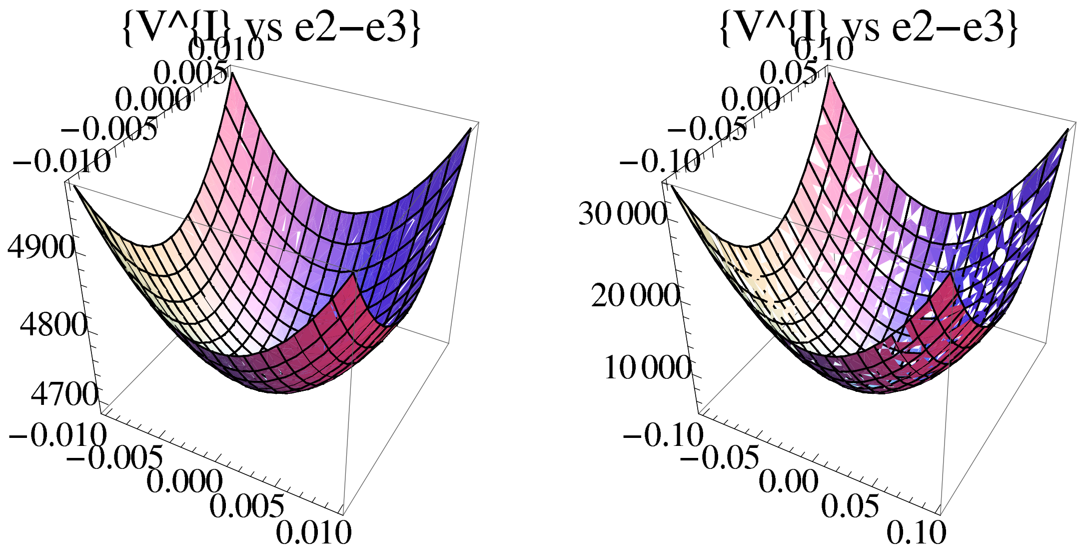

Figure B.3: *'type-1' Potential*

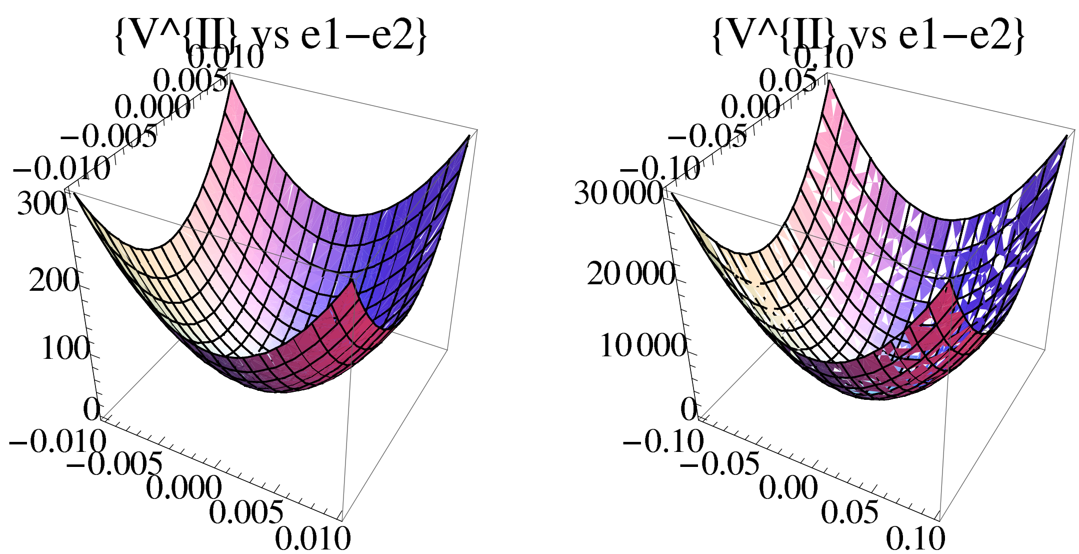

Figure B.4: *'type-2' Potential*

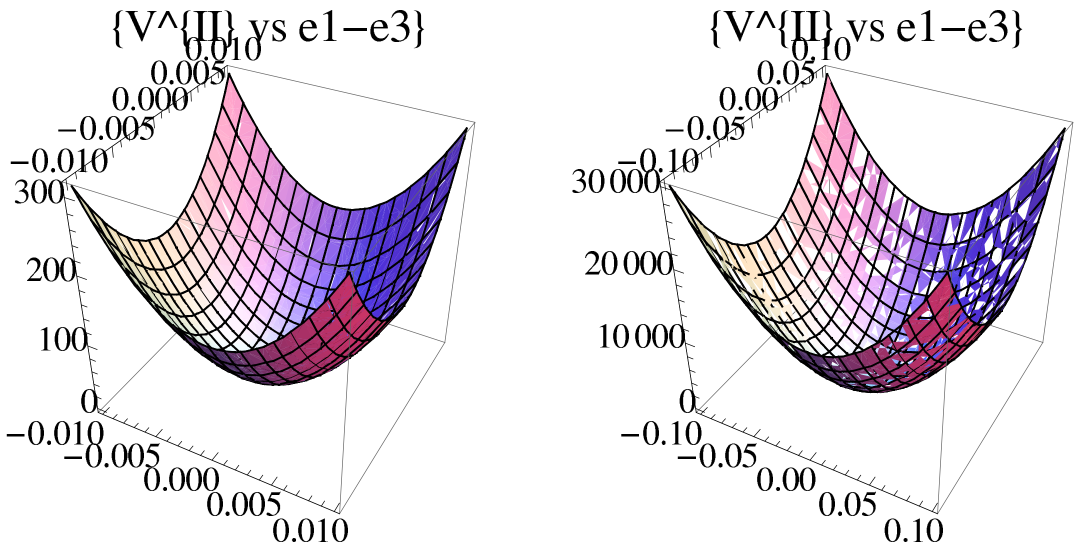

Figure B.5: *'type-2' Potential*

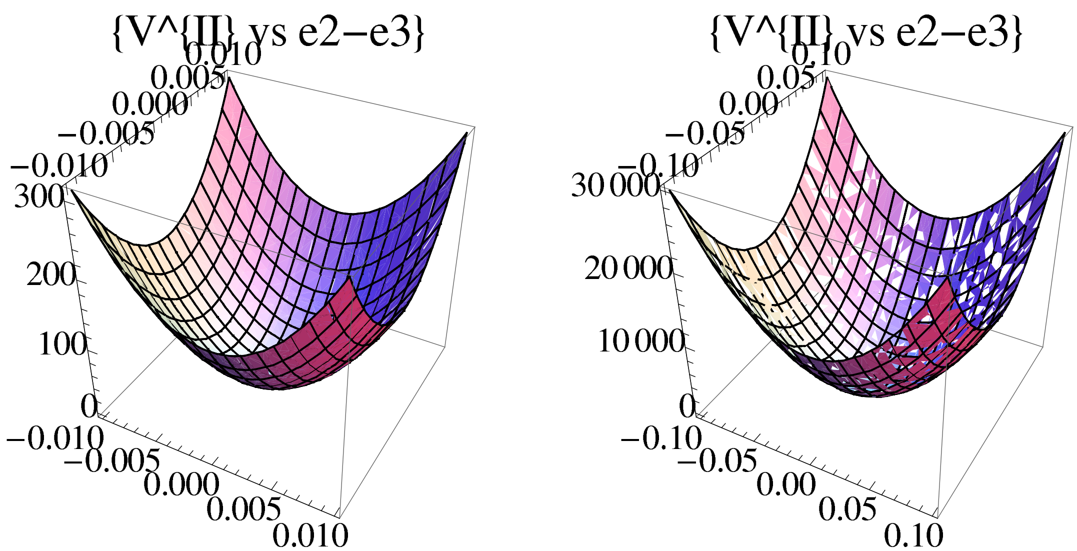

Figure B.6: *'type-2' Potential*

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List of Figures

List of Tables